Potential games and transportation models

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Potential game

$\Gamma =< N = 1, 2, ..., n, \{X_i\}_{i \in N}, \{H_i(x_1, ..., x_n)\}_{i \in N} >$
$x = (x_1, ..., x_n)$ - profile of strategies.
Objective of player $i \in N$ $H_i(x_1, ..., x_n) \to \max\limits_{x_i}$ or $\min\limits_{x_i}$
We use notation $x = (x_i, x_{-i})$ for $x = (x_1, ..., x_i, ..., x_n)$.
Definition 1. Profile $x^* = (x_1^*, ..., x_n^*)$ is Nash equilibrium if for any $i \in N$

$$H_i(x_i, x_{-i}^*) \leq H_i(x^*), \quad \forall x_i.$$

Potential game (L.S. Shapley and D. Monderer [1996])

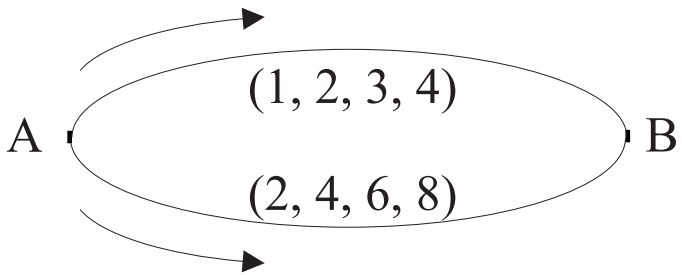
A normal-form n-player game $\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >$.

Suppose that there exists a certain function $P : \prod_{i=1}^{n} X_i \to R$ such that for any $i \in N$ we have the inequality

$$H_i(x_{-i}, x_i') - H_i(x_{-i}, x_i) = P(x_{-i}, x_i') - P(x_{-i}, x_i)$$

for arbitrary $x_{-i} \in \prod_{j \neq i} X_j$ and any strategies $x_i, x_i' \in X_i$. Then Γ is potential game and P is a potential function.

Potential games



Traffic jamming. Suppose that players *I* and *II*, each possessing two packages, have to deliver it from point A to point B.

These points communicate through two links. Numbers on the figure indicate the journey time on each link depending on the number of moving packages.

Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-8,-8) & (-6,-5) & (-4,-8) \\ (-5,-6) & (-6,-6) & (-7,-12) \\ (-8,-4) & (-12,-7) & (-16,-16) \end{pmatrix} \end{array}.$$

Potential games

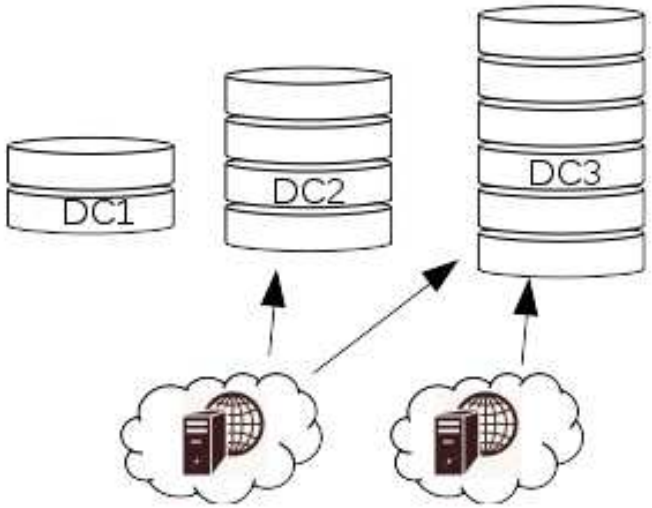
Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-8,-8) & (-6,-5) & (-4,-8) \\ (-5,-6) & (-6,-6) & (-7,-12) \\ (-8,-4) & (-12,-7) & (-16,-16) \end{pmatrix} \end{array}.$$

The game possesses the potential

$$P = \begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} 13 & 16 & 13 \\ 16 & 16 & 10 \\ 13 & 10 & 0 \end{pmatrix} \end{array}$$

Potential games



Choice of data centers. Assume each of two cloud operators may conclude a contract to utilize the capacity resources of one or two of three data centers available. The resources of data centers 1, 2, and 3 are 2, 4, and 6, respectively. If both operators choose the same data center, they equally share its resources. The payoff of each player is the sum of the obtained resources at each segment minus the rent cost of the resources provided by a data center (let this cost be 1).

Potential games

Payoff matrix:

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\ \begin{pmatrix} (0,0) & (1,3) & (1,5) & (0,3) & (0,5) & (1,8) \\ (3,1) & (1,1) & (3,5) & (1,2) & (3,6) & (1,6) \\ (5,1) & (5,3) & (2,2) & (5,4) & (2,3) & (2,5) \\ (3,0) & (2,1) & (4,5) & (1,1) & (3,5) & (2,6) \\ (5,0) & (6,3) & (3,2) & (5,3) & (2,2) & (3,5) \\ (8,1) & (6,1) & (5,2) & (6,2) & (5,3) & (3,3) \end{pmatrix} \end{array}.$$

Potential:

$$P = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\ \begin{pmatrix} 1 & 4 & 6 & 4 & 6 & 9 \\ 4 & 4 & 8 & 5 & 9 & 9 \\ 6 & 8 & 7 & 9 & 8 & 10 \\ 4 & 5 & 9 & 5 & 9 & 10 \\ 6 & 9 & 8 & 9 & 8 & 11 \\ 9 & 9 & 10 & 10 & 11 & 11 \end{pmatrix} \end{array}.$$

Potential games

Theorem. *Let an n-player game* $\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >$ *have a potential* P*. Then a Nash equilibrium in the game* Γ *represents a Nash equilibrium in the game* $\Gamma' =< N, \{X_i\}_{i \in N}, P >$*, and vice versa. Furthermore, the game* Γ *admits at least one pure strategy equilibrium.*

Proof. The first assertion follows from the definition of a potential.

$$H_i(x^*_{-i}, x_i) \leq H_i(x^*), \forall x_i, \quad P(x^*_{-i}, x_i) \leq P(x^*), \forall x_i$$

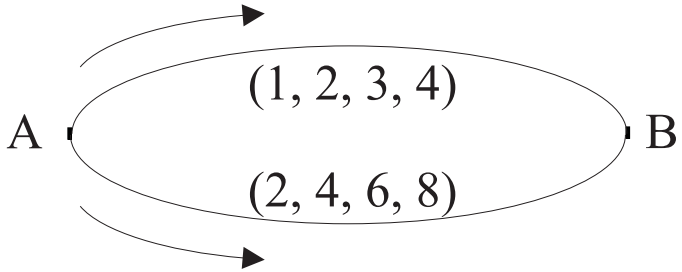
Potential games

Now, we argue that the game Γ' always has a pure strategy equilibrium. Let x^* be the pure strategy profile maximizing the potential $P(x)$ on the set $\prod_{i=1}^{n} X_i$. For any $x \in \prod_{i=1}^{n} X_i$, the inequality $P(x) \leq P(x^*)$ holds true at this point, particularly,

$$P(x^*_{-i}, x_i) \leq P(x^*), \forall x_i.$$

Therefore, x^* represents a Nash equilibrium in the game Γ' and, hence, in the game Γ.

Potential games



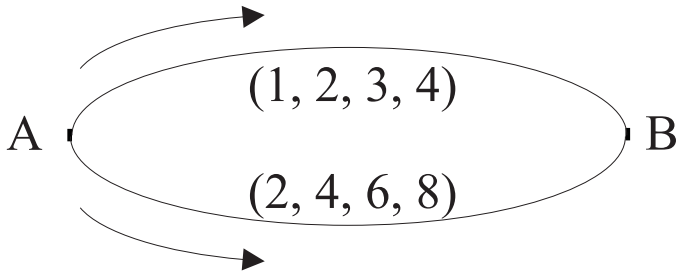
A game without potential. A game may have no potential, even if a pure strategy equilibrium does exist.

Suppose that the costs of players are defined by the maximal journey time of their packages on both links.

Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-4,-4) & (-3,-3) & (-2,-4) \\ (-3,-3) & (-4,-4) & (-6,-6) \\ (-4,-2) & (-6,-6) & (-8,-8) \end{pmatrix} \end{array}.$$

Potential games



The described game has no potential. We demonstrate this fact rigorously. Assume that a potential P exists; then:

$$P(1,1) - P(3,1) = H_1(1,1) - H_1(3,1) = -4 - (-4) = 0,$$

$$P(1,1) - P(1,2) = H_2(1,1) - H_2(1,2) = -4 - (-3) = -1.$$

And so,

$$P(3,1) - P(1,2) = -1.$$

On the other hand,

$$P(1,2) - P(3,2) = H_1(1,2) - H_1(3,2) = -3 - (-6) = 3,$$

$$P(3,1) - P(3,2) = H_2(3,1) - H_2(3,2) = -2 - (-6) = 4,$$

whence it follows that $P(3,1) - P(1,2) = 1$. This two facts contradicts, the game possesses no potential.

Gongestion games

Definition. *A symmetrical congestion game is an n-player game* $\Gamma =< N, M, \{S_i\}_{i \in N}, \{c_i\}_{i \in N} >$, *where* $N = \{1, ..., n\}$ *stands for the set of players, and* $M = \{1, ..., m\}$ *means the set of some objects for strategy formation. A strategy of player* i *is the choice of a certain subset from* M. *The set of all feasible strategies makes the strategy set of player* i, *denoted by* S_i, $i = 1, ..., n$. *Each object* $j \in M$ *is associated with a function* $c_j(k), 1 \leq k \leq n$, *which represents the payoff (or costs) of each player from* k *players that have selected strategies containing* j. *This function depends only on the total number* k *of such players.*

Gongestion games

Players have chosen strategies $s = (s_1, ..., s_n)$. The payoff function of player i is determined by the total payoff on each object:

$$H_i(s_1, ..., s_n) = \sum_{j \in s_i} c_j(k_j(s_1, ..., s_n)).$$

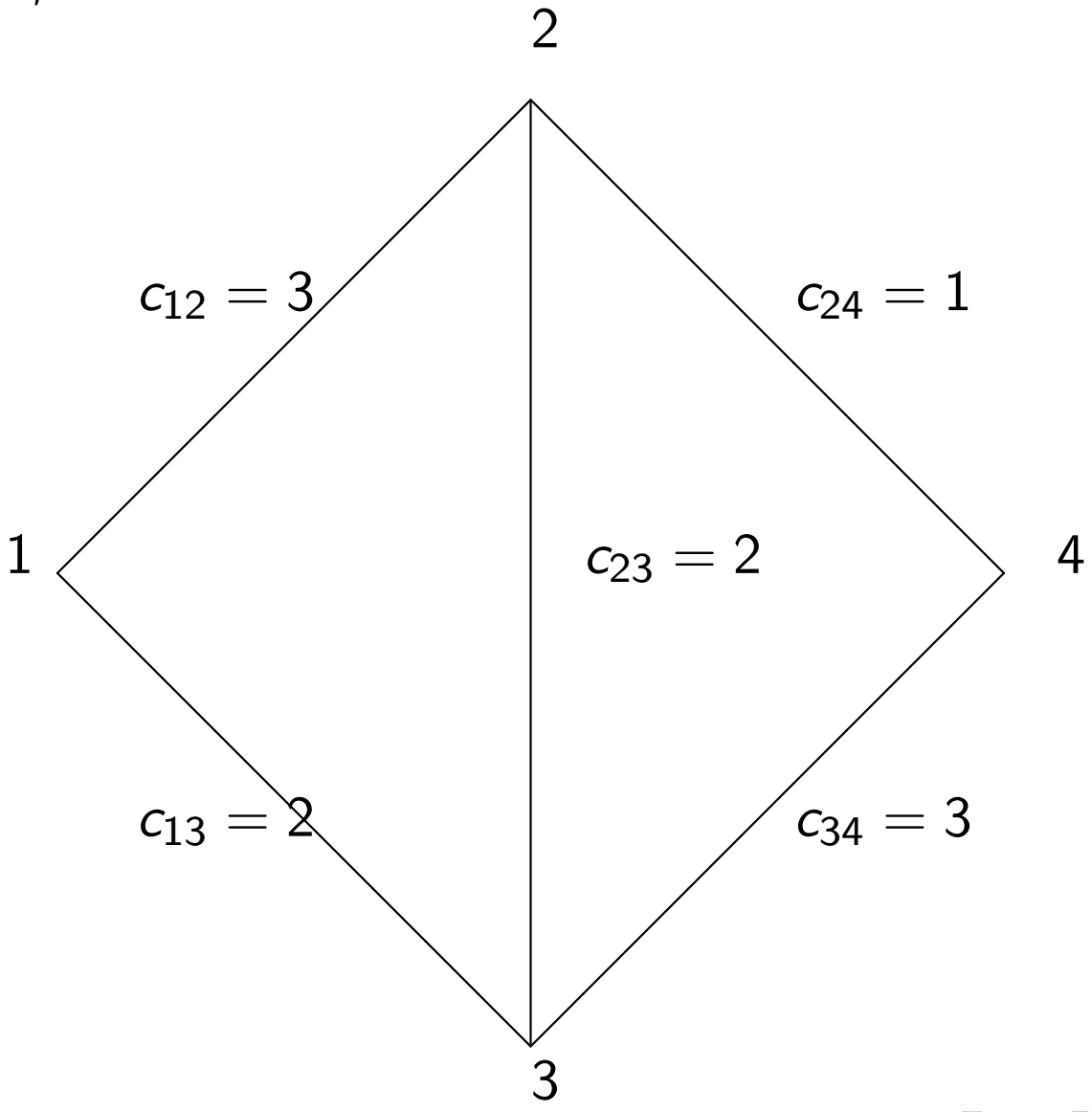
Here $k_j(s_1, ..., s_n)$ gives the number of players whose strategies incorporate object j, $i = 1, ..., n$.

Theorem. *A symmetrical congestion game is potential, ergo admits a pure strategy equilibrium.*

$$P(s_1, ..., s_n) = \sum_{j \in \cup_{i \in N} s_i} \left(\sum_{k=1}^{k_j(s_1,...,s_n)} c_j(k) \right)$$

Gongestion game. Example

Players 1,2,3,4 send unit traffic via network from 1 to 4



Gongestion game. Example

Strategies: $s_1 = \{(1,2)(2,4)\}$, $s_2 = \{(1,3)(3,4)\}$,
$s_3 = \{(1,2)(2,3),(3,4)\}$, $s_4 = \{(1,3)(3,2),(2,4)\}$.
Calculate potential:

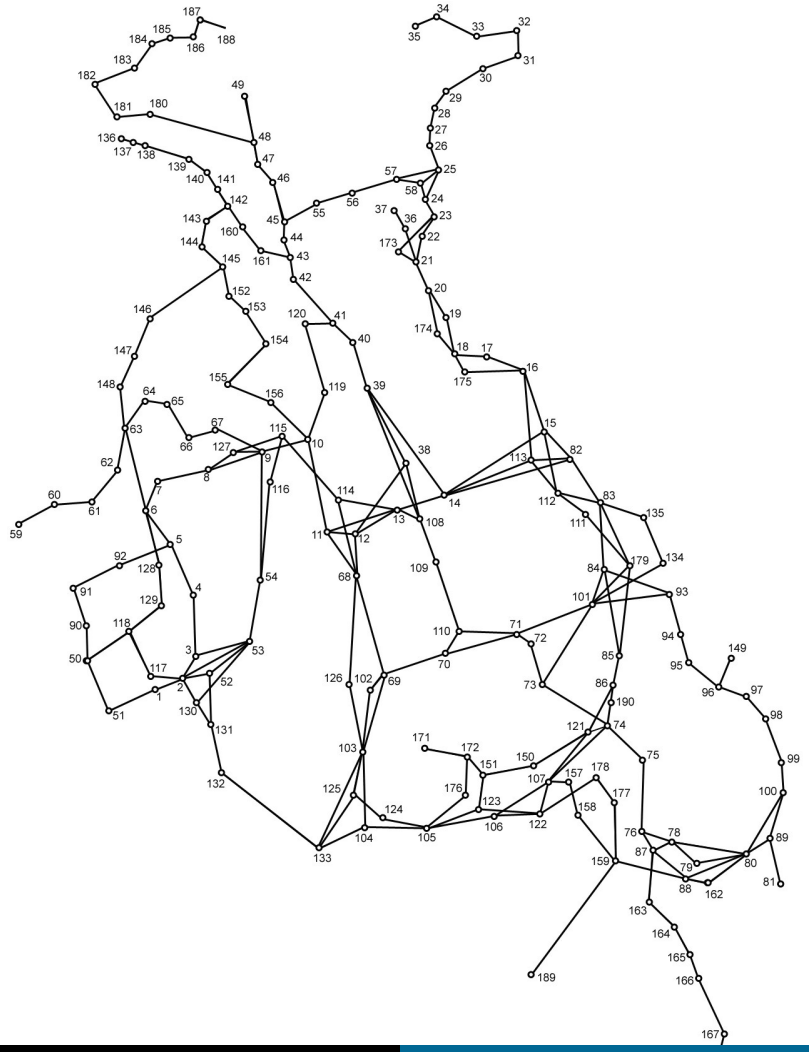
$$P(s_1,s_2,s_3,s_4) = \frac{1+2}{3} + \frac{1+2}{1} + \frac{1+2}{2} + \frac{1+2}{2} + \frac{1+2}{3} = 8$$
$$P(s_1,s_1,s_2,s_2) = \frac{1+2}{3} + \frac{1+2}{1} + \frac{1+2}{2} + \frac{1+2}{2} + \frac{1+2}{3} = 6.5$$
$$P(s_1,s_2,s_2,s_3) = \frac{1+2}{3} + \frac{1}{1} + \frac{1+2}{2} + \frac{1}{2} + \frac{1+2+3}{3} = 6.$$

Thus, profile (s_1,s_2,s_2,s_3) is Nash equilibrium.

Transportation network and correspondence matrix

A typical example of an urban transportation network is shown in Figure. In the general case, a transportation model can be represented by a graph with nodes acting as stops and edges as transport passageways

In transportation problems, an important role is played by the correspondence matrix, which characterizes the intensity of passenger traffic between different stops. We will adopt the Poisson flow model.



Correspondence matrix

Assume there are direct passenger traffic flows on this route. Denote by $x_1^r,\ x_2^r,\ ...,\ x_K^r$ the numbers of incoming passengers at stops $i = 1, ..., K$, respectively, in experiment r, where $r = 1, ..., N$. By analogy, the values $y_1^r,\ y_2^r,\ ...,\ y_K^r$ will characterize the number of outgoing passengers at stops $j = 1, ..., K$, respectively, in experiment r, where $r = 1, ..., N$. Obviously, $x_K^r = y_1^r = 0$ for any experiment r. The values x_i^r and y_j^r are observable in the model. Designate as λ_{ij}^r the number of passengers incoming at stop i and outgoing at stop j. These are unobservable values that have to be estimated.

Correspondence matrix

$y_{11}^r = 0$	y_{12}^r	y_{13}^r	$\dots$	y_{1K-1}^r	y_{1K}^r	x_1^r
0	$y_{22}^r = 0$	y_{23}^r	$\dots$	y_{2K-2}^r	y_{2K}^r	x_2^r
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
0	0	0	$\dots$	$y_{K-1,K-1}^r = 0$	$y_{K-1,K}^r$	x_{K-1}^r
0	0	0	$\dots$	0	0	$x_K^r = 0$
$y_1^r = 0$	y_2^r	y_3^r	$\dots$	y_{K-1}^r	y_K^r	

Table demonstrates the results of experiment r. All elements in the last column are the sums of elements from a corresponding row; similarly, the last row contains the sums of elements from a corresponding column.

Correspondence matrix

Denote by p_{ij} the share of passengers who get into the bus at stop i and get out at stop j.
lin a rough approximation we have the equality

$$y_j^r \approx \sum_{i=1}^{j-1} x_i^r p_{ij}.$$

To find the unknown variables p_{ij} $(i < j, j = 1, ..., K)$, we employ **the least-squares method**. It is necessary to minimize the sum of squared deviations

$$s = \sum_{r=1}^{N} \sum_{j=2}^{K} (y_j^r - \sum_{i=1}^{j-1} x_i^r p_{ij})^2$$

subject to the constraints

$$\sum_{j=i+1}^{K} p_{ij} = 1, i = 1, , K-1, \quad p_{ij} \geq 0, \forall i, j.$$

Correspondence matrix

After obtaining the estimates p_{ij}, we calculate the passenger traffic intensities between nodes i and j by the formula

$$\lambda_{ij} = \frac{1}{N} \sum_{r=1}^{N} x_i^r p_{ij}, i, j = 1, 2, ..., K.$$

A model of public transport system

An aspect of public transport modeling consists in calculation of passenger service characteristics such as pricing, the required number of vehicles, their capacity and schedule (routes and traffic intervals or intensities).

Service rate. The system with two carriers

Let λ be the intensity of passenger traffic on a certain route. Assume two carriers (players) deliver passengers on the route; their strategies are service rates μ_1 and μ_2. Denote by p a given fare for transportation. The transportation cost of both carriers is proportional to the volumes of passenger traffic with coefficients c_1 and c_2, respectively.

Under chosen service rates, the first carrier delivers $\lambda\mu_1/(\mu_1+\mu_2)$ passengers per unit time, and its income makes up

$$H_1(\mu_1,\mu_2) = p\lambda\frac{\mu_1}{\mu_1+\mu_2} - c_1\mu_1. \tag{1}$$

Similarly, the income of the second carrier is

$$H_2(\mu_1,\mu_2) = p\lambda\frac{\mu_2}{\mu_1+\mu_2} - c_2\mu_2. \tag{2}$$

The payoff functions (1), (2) are concave and hence this game always possesses a Nash equilibrium.

The first-order optimality conditions $\partial H_i/\partial \mu_i = 0, i = 1, 2$, yield

$$p\mu_2 = c_1(\mu_1 + \mu_2)^2, \quad p\mu_1 = c_2(\mu_1 + \mu_2)^2,$$

and the Nash equilibrium has the form

$$\mu_1^* = p\lambda \frac{c_2}{(c_1 + c_2)^2}, \quad \mu_2^* = p\lambda \frac{c_1}{(c_1 + c_2)^2}.$$

Service rate. The system with K carriers

In the general case, a transportation network is described by a graph with N stops served by K carriers.
Denote by L_k the number of routes used by carrier k, $(k = 1, \ ..., \ K)$ and by λ_{ij} the intensity of a Poisson flow of the passengers that come to stop i with a view of moving to stop j.

Assume that $\lambda_{ij} \geq 0$, $\lambda_{ii} = 0$, $i, j = 1, \ ..., \ N$. As before, p is the fare for public transportation; in addition, for carrier k let c_{kl} specify the cost of a single trip on route l, where $l = 1, \ ..., \ L_k$, $k = 1, \ ..., \ K$.

We will write $\delta_k^l(ij) = 1$ if it is possible to move between stops i and j on route l using the service of carrier k and $\delta_k^l(ij) = 0$ otherwise $(i, j = 1, \ ..., \ N, l = 1, \ ..., \ L_k, \ k = 1, \ ..., \ K)$.

Service rate equilibrium

Define the strategies of the players as μ_{kl}, which is the intensity of bus traffic (service rate) of carrier k on route l ($l = 1, \ldots, L_k, k = 1, \ldots, K$). Obviously, $\mu_{kl} \geq 0$ for $l = 1, \ldots, L_k$, $k = 1, \ldots, K$. Denote by μ the strategy profile.

The passenger flow of the intensity λ_{ij} can be served by several carriers operating on different routes. Therefore, the total service rate of the carriers on these routes must exceed λ_{ij}, i.e.,

$$\sum_{k=1}^{K} \sum_{l=1}^{L_m} \delta_k^l(ij)\mu_{kl} \geq \lambda_{ij}, \forall i, j. \tag{3}$$

At each stop, the passengers get into a first bus with enough empty places that can move them to a desired stop. The other passengers wait for a next bus with enough empty places.

Sevice rate equilibrium

The passenger traffic flow is distributed among the carriers proportionally to their service rates on a given route, i.e., the average number of passengers delivered by carrier k on route l per unit time has the form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}\cdot\frac{\delta_k^l(ij)\mu_{kl}}{\sum_{m=1}^{K}\sum_{r=1}^{L_m}\delta_m^r(ij)\mu_{mr}},\quad l=1,\ \ldots,\ L_k, k=1,\ \ldots,\ K.$$

The profit of carrier k is its income (the fares of all passengers) minus the transportation cost per unit time, i.e.,

$$H_k(\mu)=\sum_{i=1}^{N}\sum_{j=1}^{N}p\cdot\sum_{l=1}^{L_k}\lambda_{ij}\cdot\frac{\delta_k^l(ij)\mu_{kl}}{\sum_{m=1}^{K}\sum_{r=1}^{L_m}\delta_m^r(ij)\mu_{mr}}-\sum_{l=1}^{L_k}\alpha_{kl}\mu_{kl},\quad k=1,\ \ldots$$

In addition, the transportation cost must not exceed the maximum possible income from service, that is,

$$p\sum\sum\lambda_{ij}\delta_k^l(ij)\geq\sum^{L_k}c_{kl}\mu_{kl}, k=1,\ \ldots,\ K.$$

Wardrop equilibrium principle

In 1952 Wardrop suggested the principle of equilibrium flows in the following statement: any transportation system reaches an equilibrium state after some period of time [Wardrop 1952]. According to the Wardrop principle, the trip time along all existing routes is the same for all road users and smaller than the trip time of any road user deviating from his route. The average trip time achieves minimum in the equilibrium. It is possible to show that a Wardrop equilibrium represents a Nash equilibrium in a game with transport vehicles as players and routes as their strategies. Interestingly, this game is a congestion game with a potential. Hence, equilibrium calculation comes to potential minimization.

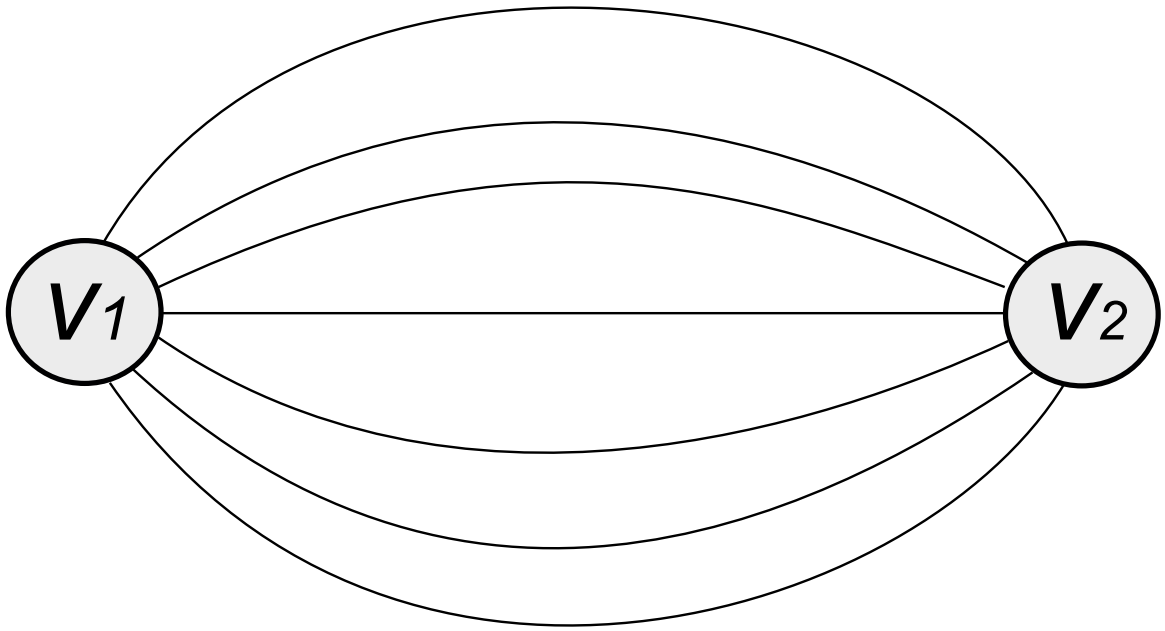
Latency function

Assume the incoming flow of players is described by a Poisson process X of intensity x and delay is determined via latency function $f(x)$. For example in transportation models very often is used BPR latency function given by

$$f_i(x) = t_i \left(1 + \alpha_i \left(\frac{x}{c_i}\right)^{\beta_i}\right), i = 1, 2, ..., n,$$

where t_i indicates the trip time in the unoccupied channel i, c_i specifies the capacity of channel i, and the constants α_i, β_i are determined by road parameters (width, the number of traffic control devices, e.g., light signals, etc.). Note that these constants can be adjusted using statistical data.

N parallel links



The incoming flow X is decomposed into n subflows running through corresponding channels. Designate as $x_i, i = 1, ..., n$ the values of the subflows. It is clear that $x_i \geq 0$ and

$$\sum_{i=1}^{n} x_i = X. \tag{4}$$

Denote by $x = (x_1, ..., x_n)$ the subflow profile.

Wardrop equilibrium

The game with payoff functions

$$H_i(x) = f_i(x), i = 1, ..., n$$

is a potential game with potential function

$$P(x) = \sum_{i=1}^{n} \int_0^{x_i} f_i(u)du.$$

Wardrop equilibrium

Consider the parallel links and BPR latency function case. Let

$t_1 \leq t_2 \leq ... \leq t_n$.

The equilibrium can be calculated by solving the minimization problem with the goal function

$$P(x) = \sum_{i=1}^{n} \int_0^{x_i} t_i \left(1 + \alpha_i \left(\frac{u}{c_i}\right)^{\beta_i}\right) du$$

and the constraints

$$\sum_{i=1}^{n} x_i = X,$$

$$x_i \geq 0, \forall i = 1, \ ..., \ n.$$

Wardrop equilibrium

Karush–Kuhn–Tucker theorem is applied. Construct the Lagrange function

$$L(x, \lambda) = P(x) + \lambda(X - \sum_{i=1}^{n} x_i) + \sum_{i=1}^{n} \lambda_i(-x_i)$$

and apply the first-order optimality conditions with respect to x_i to get

$$t_i \left(1 + \alpha_i \left(\frac{x_i}{c_i}\right)^{\beta_i}\right) - \lambda_i = \lambda, \quad i = 1, ..., n.$$

Theorem

For the equilibrium flow to be distributed among the first k channels, a necessary and sufficient condition is

$$V_{k-1} < X \leq V_k, \quad k = 1, 2, ..., n.$$

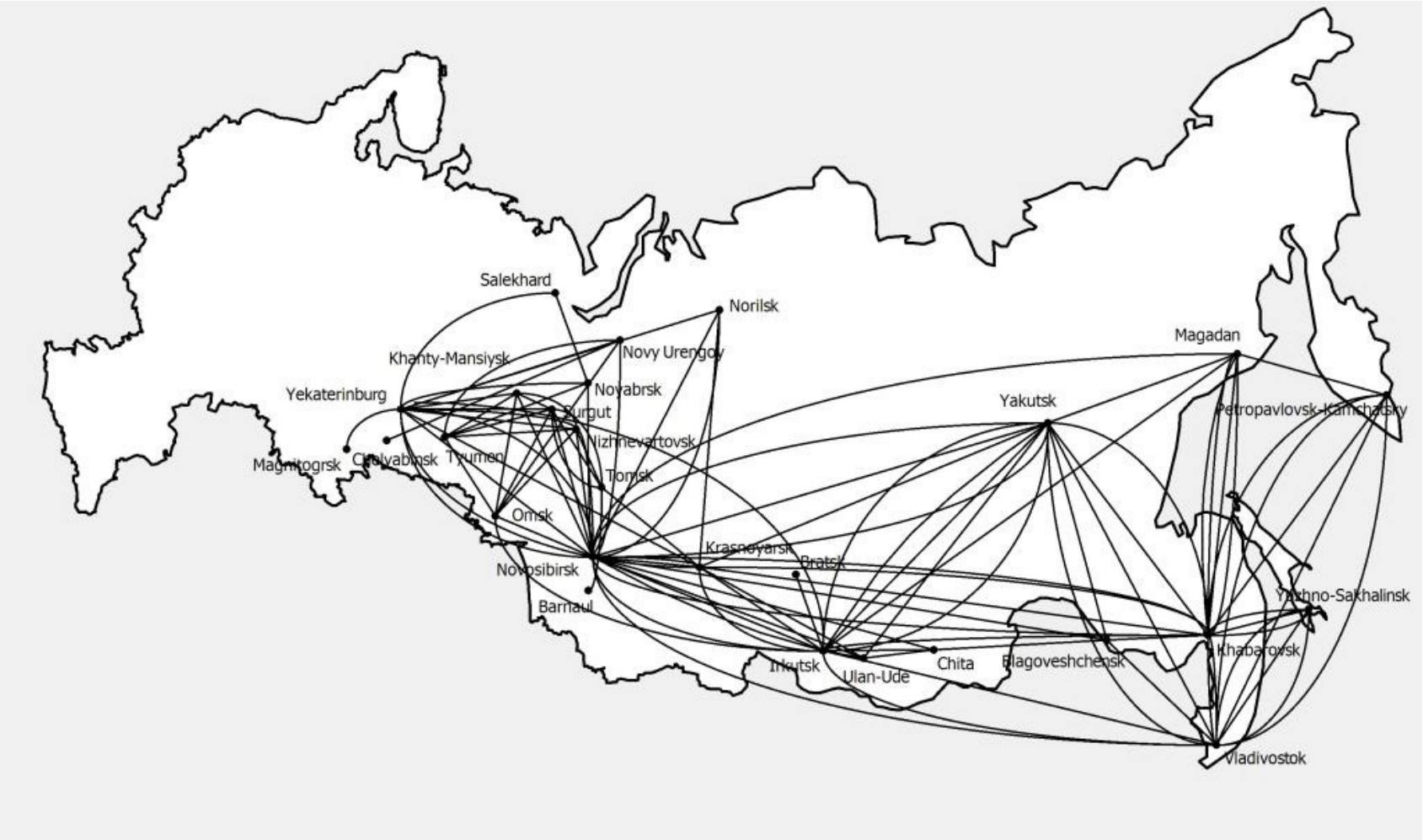
Here $V_k = \sum_{i=1}^{k} x'_{i,k+1}, \quad k = 1, 2, ..., n,$ *and* x'_{ij} *satisfies the combined equations*

$$f_i(x) = t_j, \quad j = i + 1, ..., n.$$

Moreover, the equilibrium distribution x_{eq} *is the solution of the combined equations*

$$\begin{cases} x_1 + x_2 + ... + x_k = X, \\ f_1(x_1) = f_2(x_2) = ... = f_k(x_k). \end{cases}$$

Models of Transportation Market



Russian air transportation market

Market description

The customers are distributed in the nodes of a transportation graph $G(V, E)$.
The nodes of a graph $G(V, E)$ represent the hubs (bus stops, airports, railway stations, etc.) while its edges correspond to the transportation links (railways, car, air lines, etc.).

The customers in a node v_j are the passengers who use this kind of transportation.
Service is delivered only if there exists a link between customers, i.e., an edge $e_j \in E$ in the graph $G(V, E)$.
The number of customers in a node $v_j \in V$ – capacity. The demand depends on the capacity of nodes connected by the edge e_j, i.e., on the number of customers in these nodes:

$$d(e_j) = d(v_1, v_2), \qquad e_j = (v_1, v_2).$$

Market description

There are n companies (players) delivering a service in the market. Assume player i has m_i units of a resource. He distributes the resource among links of the graph $G(V, E)$.
By distributing the resource to a link e_j, a player connects the corresponding nodes and then the customers may use the resource of player i. Therefore, each of the players forms his own transportation network E^i, which is a subset of the links in the graph $G(V, E)$.
A competition occurs in the market only if a link e_j is included into several transportation networks simultaneously, i.e.,

$$\exists\, i, j: \; E^i \cap E^j \neq \varnothing, \quad i, j \in N, \; i \neq j.$$

Pricing model

The players announce their prices for the service on e_j, thereby competing with each other. For the demand $d(e_j)$, the share M_{ij} of the customers who prefer the service of player i depends on the price p_{ij} and the prices of other players on this link:

$$M_{ij} = M_{ij}(p_{ij}, \{p_{rj}\}_{r \in N_j \setminus \{i\}}),$$

where N_j is the set of competing players on e_j.
The number of customers who prefer the service of player i on a link e_j is

$$S_{ij}(\{p_{rj}\}_{r \in N_j}) = M_{ij}(p_{ij}, \{p_{rj}\}_{r \in N_j \setminus \{i\}}) d(e_j).$$

Pricing model

Denote by x_{ij} a distribution of player i on a link e_j, that is,

$$x_{ij} = \begin{cases} 1 & \text{if} \quad e_j \in E^i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the number of players on a link e_j is

$$|N_j| = \sum_{i=1}^{n} x_{ij}.$$

Player i with m_i units of the resource over the graph $G(V, E)$ attracts the number of customers

$$S_i = \sum_{j=1}^{|E|} M_{ij}(p_{ij}, \{p_{rj}\}_{r \in N_j \setminus \{i\}}) d(e_j) x_{ij}.$$

Payoffs

The income of player i on a link e_j depends on the price for the service and his share in the customer demand:

$$h_{ij}(\{p_{rj}\}_{r\in N_j}) = p_{ij}M_{ij}(p_{ij}, \{p_{rj}\}_{r\in N_j\setminus\{i\}})d(e_j).$$

Supplying a unit of the resource on a link e_j, player i bears expenses c_{ij}. Assume the cost of the resource is proportional to the number of customers who use it. Then the payoff of player i over the graph $G(V, E)$ has the form

$$H_i(\{p_r\}_{r\in N}, \{x_r\}_{r\in N}) = \sum_{j=1}^{|E|}\big(h_{ij}(p_{ij}, \{p_{rj}\}_{r\in N_j\setminus\{i\}}) - \\ - c_{ij}S_{ij}(p_{ij}, \{p_{rj}\}_{r\in N_j\setminus\{i\}})\big)x_{ij}.$$

Three-stage game

The game has three stages as follows.

1. The players simultaneously distribute their resources using allocations $\{x_i\}_{i \in N}$.
2. The players simultaneously announce the prices for their resources $\{p_i\}_{i \in N}$.
3. The customers choose a preferable service and the players obtain the payoffs $\{H_i\}_{i \in N}$ depending on their transportation networks and prices.

Logistic approach

Logistic distribution [McFadden 1973].

$$M_{ij} = \frac{e^{a_1 p_{ij} + (a, v_i)}}{\sum_{s=1}^{|N_j|} e^{a_1 p_{sj} + (a, v_s)} + e^{\rho}}, \quad e_j \in E^i, \tag{5}$$

where v_i is a vector of characteristics (observable factors) for the service of player i except the price.
A customer may refuse of the resource on a link e_j if the players overprice their services. The model contains the term e^{ρ} in the denominator of (**??**), which describes the customers who prefer not to use any service at all on the link e_j.

Theorem

Pricing game with payoff functions $H_{ij}(p_{ij})$ is a potential game.

Logistic approach

Theorem

In the pricing game with additional player $\tilde{\Gamma}^j_G$*, the equilibrium prices are decreasing for all players except the new one, i.e,* $\forall\, i \in N_j : \;\; p^*_{ij} > \tilde{p}^*_{ij}$.

Corollary

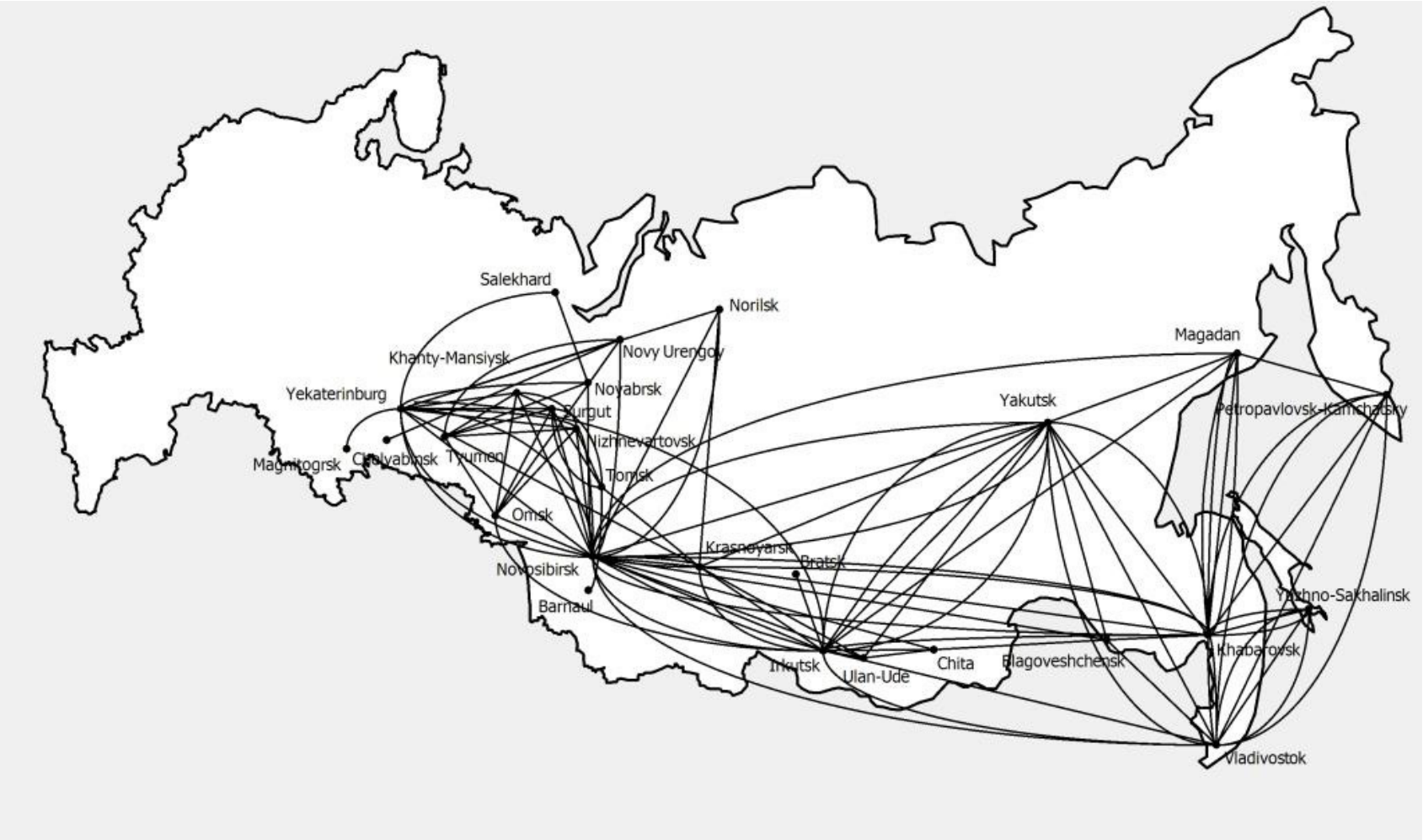
In the pricing game with additional player $\tilde{\Gamma}^j_G$*, the optimal payoffs are decreasing for all players except the new one, i.e.,* $\forall\, i \in N_j$:

$$h^p_{ij}\big(\{p^*_{ij}\}_{i\in N_j}\big) > h^p_{ij}\big(\{\tilde{p}^*_{ij}\}_{i\in \tilde{N}_j}\big).$$

Modeling of air transportation market

The players are the airlines operating in this market, which seek to increase the profit from passenger carriage. Each airline delivers its air transportation service over its transportation network, which consists of airports and air routes between them.
A flight performed by each airline has certain characteristics affecting the choice of an appropriate airline for the passengers. For example, these are departure-arrival time, flight time, aircraft type, departure-destination airports (for megalopolises), etc.
Two airports can be connected by the flights of several airlines. In this case, the airlines compete with each other in order to attract as much passengers as possible for profit maximization. Although the choice of a passenger is conditioned by many factors, we will consider the price-based competition of the airlines.

Russian air transportation market



Summary statistics for Russian market

Factor	Mean	σ	Median	Min	Max
Price (RUB)	9831	4111	9425	1500	21630
Flight time (hours)	3.34	2.33	2.4	0.4	15.3
Frequency (flights per week)	2.8	2.04	1	1	14
Distance (km)	1774	1263	1486	215	7314
Annual income (RUB)	27 053	10 390	22 224	14 167	50 991
Population	499 430	394 948	327 423	44 334	1 498 9

Airlines in the market

The Russian market contains 27 airports with 95 routes in the graph $G(V, E)$. There exist 239 direct flights and 74 transfer flights (with connections). The number of airlines is 11 and the maximum number of competitive airlines on a single route is 5.

Airline	Airports	Routes
Aurora (former Vladivostok Avia)	10	11
Yakutia	10	11
SAT Airlines	4	3
IrAero	13	14
S7 Airlines	11	14
Ural Airlines	9	9
Angara	5	4
Tomsk Avia	4	3
NordStar	11	10
RusLine	9	8
UTAir	12	21

Airports in Russian market (fragment)

Airport (city, IATA code)	Population	Airlines
Yuzhno-Sakhalinsk (UUS)	186 267	3
Petropavlovsk-Kamchatsky (PKC)	179 784	4
Vladivostok (VVO)	597 476	6
Magadan (GDX)	95 463	5
Khabarovsk (KHV)	585 556	7
Blagoveshchensk (BQS)	215 736	4
Yakutsk (YKS)	278 406	6
Chita (HTA)	327 423	3
Ulan-Ude (UUD)	411 646	4
Irkutsk (IKT)	597 846	8
Bratsk (BTK)	243 926	1
Krasnoyarsk (KJA)	997 316	6
Novosibirsk (OVB)	1 498 921	10
Norilsk (NSK)	177 273	3
Yekaterinburg (SVX)	1 377 738	4

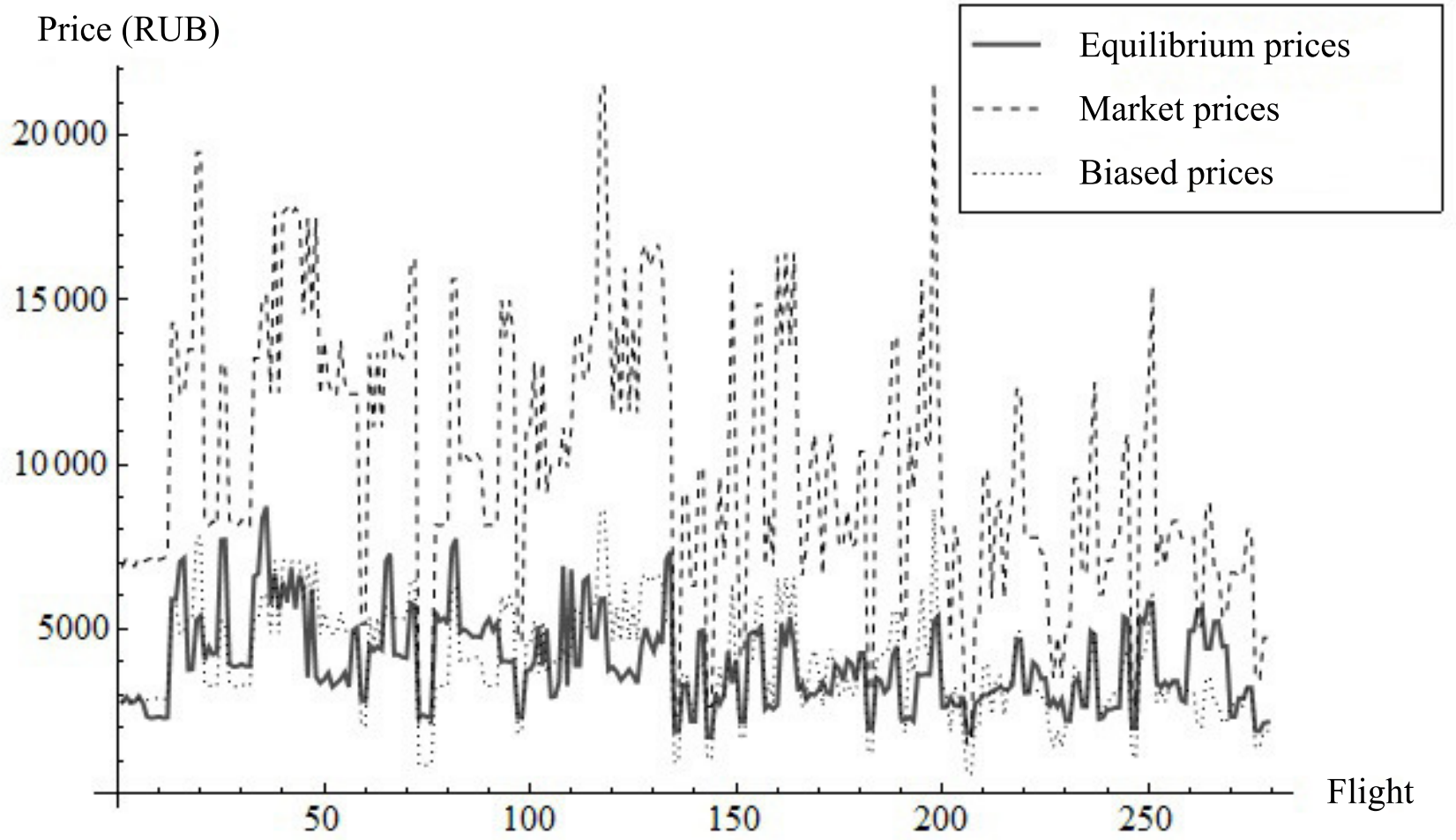
Flight characteristics for Irkutsk–Novosibirsk route

Airline	Flight time (hours)	Flights per week	Distance (km)	Connections
S7 Airlines	2.4	4	1462.6	0
IrAero	3.55	5	1520.918	1
Angara	2.1	3	1462.6	0
RusLine	2.4	3	1462.6	0
NordStar	5.2	3	1520.918	1

Flight equilibrium for Irkutsk–Novosibirsk route

Airline	Price (RUB)	Income (million RUB)	Cost (million RUB)	Profit (mullion RUB)	Share in passenger traffic
S7 Airlines	3029.95	99.58	34.95	64.63	0.23
IrAero	2986.04	38.82	16.74	22.08	0.1
Angara	3347.28	39.16	16.9	22.26	0.2
RusLine	3115.01	24.3	9.2	15.1	0.21
NordStar	2854.08	20.48	8.73	11.75	0.07

Equilibrium prices (Russian market)

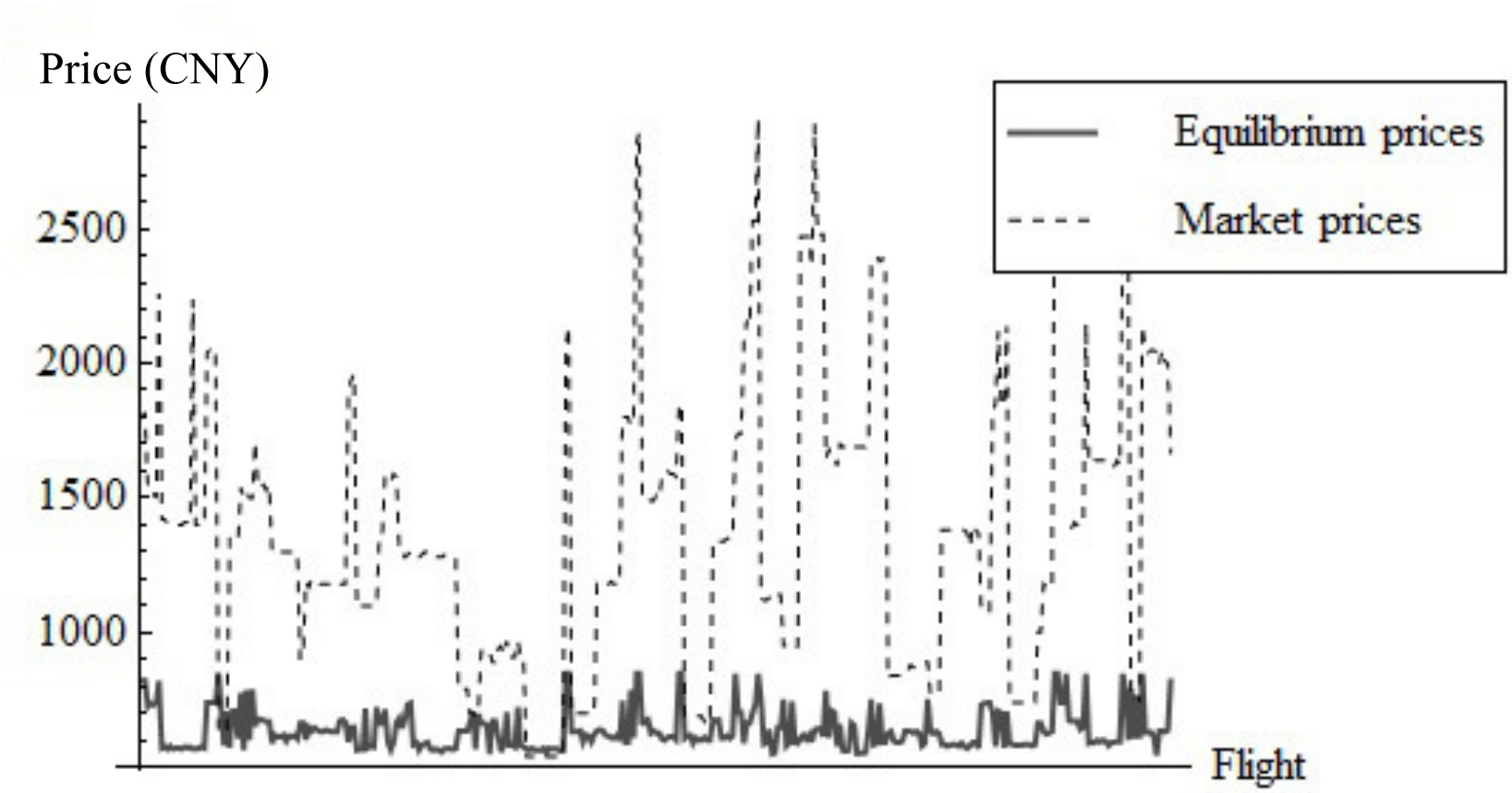


China air transportation market

Summary statistics for Chinese market

Factor	Mean	σ	Median	Min	Max
Price (CNY)	1366	537	1300	540	2910
Flight time (hours)	2.23	1	2.08	0.75	5.72
Frequency (flights per week)	6.4	1.44	7	1	7
Distance (km)	1298	657	1233	351	3388
Annual income (CNY)	29501	6903	28731	18400	40742
Population (thousand)	10968.7	7347.4	9325.05	2141.3	29190

China air transportation market



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