Graphs and collaboration networks

Vladimir Mazalov

Institute of Applied Mathematical Research,
Karelian Research Center RAS, Petrozavodsk, Russia
E-mail : vmazalov@krc.karelia.ru

Content

Social networks

Social networks represent a new phenomenon of our life.
The growing popularity of social networks in the Web dates back to 1995 when American portal Classmates.com was launched. This project facilitated the soon appearance of online social networks (SixDegrees, LiveJournal, LinkedIn, MySpace, Facebook, Twitter, YouTube, and others) in the early 2000s.
In Russia, the most popular networks are VKontakte and Odnoklassniki.
Social networks are visualized using social graphs. Graph theory provides main analysis tools for social networks. In particular, by calculating centrality measures for nodes and edges one may detect active participants (members) of a social network.

Graph construction for social network

Twitter is a popular online news and social networking service based on short messages called tweets.

Tweets were originally restricted to 140 characters, but on November 7, 2017, the limit was doubled to 280 characters for all languages except Japanese, Korean and Chinese.

As its attributes each tweet has author, text content and author's residence. A retweet is a tweet is which one author refers to another. In a message such reference has the form "@nameofAnotherAuthor."

An example of a tweet is

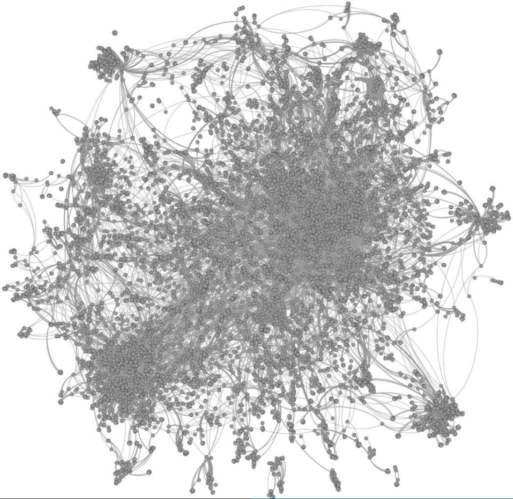
"Attended the lecture together with @VictorPetrov."

Graph of social network

Let us construct a graph of this social network.
Graph nodes are authors.
Edges correspond to retweets. If there exists a retweet of user j by user i and vice versa, we may draw an edge between nodes i and j.
Authors communicate with each other, thereby forming different communities. A community is a certain group of users with common interests (e.g., political, professional, etc.). Additional information for a structural analysis of a social network graph can be extracted from the messages of network members.

Mathematical web-portal Math-Net.ru

Thecollaboration graph from the Russian mathematical portal Math-Net.ru. The general amount of the authors is equal to 78839.



Vector space model

We will adopt the **vector space model** (VSP) to define the weights of a link (edge) in a given graph. Let each author $i, i \in N$, be associated with the so-called **bag of words**. This is a document d_i containing all words mentioned in the text of all his tweets.

Form a large bag of words from all documents of a community under study, i.e., $D = \cup_{i=1}^{n} d_i$. Assume it consists of K words. For each word $k \in d_i$ (also called term), calculate the frequency of occurrence in a document d_i and denote it by w_{ik}. This is the weight of word k in a document d_i. Then each document $d_i, i \in N$, is characterized by a K-dimensional vector $d_i = (w_{i1}, w_{i2}, ..., w_{iK})$ in space R^K. The weight of a link between two authors i and j (i.e., between documents d_i and d_j) can be defined as the cosine measure of vectors the vectors d_i and d_j, i.e.,

$$\cos(d_i, d_j) = \frac{(d_i, d_j)}{|d_i||d_j|}.$$

Vector space model

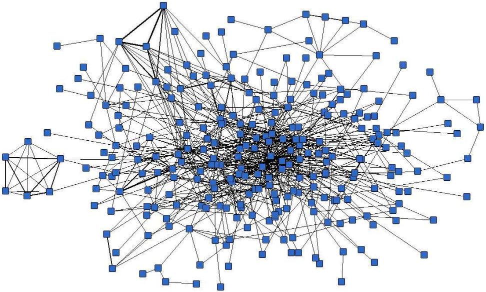
If the cosine measure is 0 or close to 0, there exists no link at all or the link is weak. Conversely, if this measure is close to 1, the link between the corresponding authors is strong.
Note that messages often contain words of little information such as prepositions, copulas, etc. Then the weight of word k in a document d_i is measured using the term frequency-inverse document frequency

$$w'_{ik} = w_{ik} + \log \frac{n}{|\{d_i \in D : k \in d_i\}|},$$

where the first summand denotes the frequency of occurrence for word k in the document d_i (as before) while the second summand characterizes how often this word is mentioned in all other documents from D. With this definition of weights, the links between two documents are measured by analogy to the previous case.

Social networks

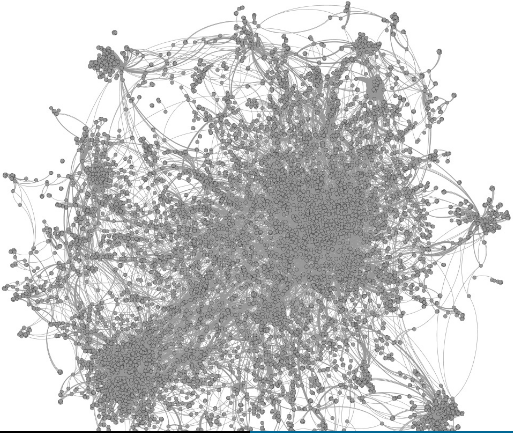
Here is the weighted graph extracted from the popular Russian social network VKontakte. The graph corresponds to the online community devoted to game theory. This community consists of 483 participants. As a weight of a link we take the number of common friends between the participants.



.: Principal component of the community Game Theory in the social network VKontakte (number of nodes: 275, number of edges: 805 and mean path's length: 3.36).

Mathematical web-portal Math-Net.ru

On fig. 2 it is presented the subgraph from the Russian mathematical portal Math-Net.ru. The general amount of the authors on the mathematical portal Math-Net.ru now is equal to 78839.



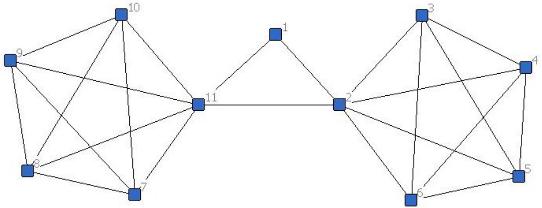
Betweenness Centrality

One of the basic concepts in the analysis of the social networks is betweenness centrality, a measure of centrality that is based on how well a node i is situated in terms of the paths that it lies on [Freeman]:

$$c_B(i) = \frac{1}{n_B} \sum_{s,t \in V} \frac{\sigma_{s,t}(i)}{\sigma_{s,t}}, \tag{1}$$

where $\sigma_{s,t}$ is the total number of geodesics (shortest paths) between nodes s and t, $\sigma_{s,t}(i)$ is the number of geodesics between s and t that i lies on. The denominator n_B captures that the node i could lie on paths between as many as $n_B = (n-1)(n-2)/2$ pairs of other nodes.

Example



.: Network of 11 nodes

$c_B(1) = 0?$

Cooperative game. Characteristic function

Consider a game where the graph g is a tree which consists on n nodes. Characteristic function is determined by the following way. Every direct connection gives to coalition S the impact r, where $0 \leq r \leq 1$.

Moreover, players obtain an impact from non-direct connections. Each path of length 2 gives to coalition S the impact r^2, a path of length 3 gives to coalition the impact r^3, etc.

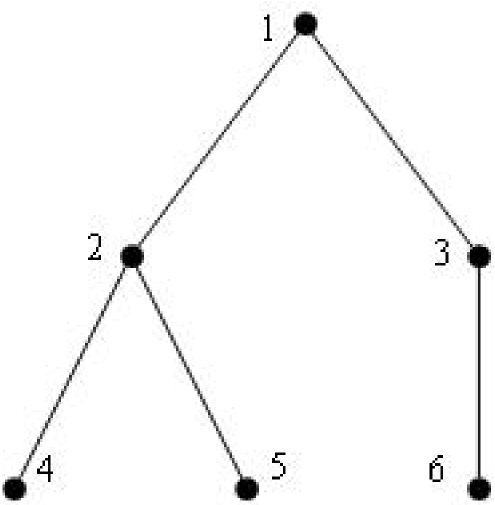
For any coalition S we obtain

$$v(S) = a_1 r + a_2 r^2 + \cdots + a_k r^k + \cdots + a_L r^L = \sum_{k=1}^{L} a_k r^k, \qquad (2)$$

where L is a maximal distance between two nodes in the coalition; a_k is the number of paths of length k in this coalition.

$$v(i) = 0, \ \forall i \in N.$$

Characteristic function. Example



For the tree on fig. 1 we find $L = 4$, $a_1 = 5$, $a_2 = 5$, $a_3 = 3$, $a_4 = 2$. Consequently, the value of grand-coalition is

$$v(N) = 5r + 5r^2 + 3r^3 + 2r^4.$$

For coalition $S = \{1, 2, 4, 5\}$ $L = 2$, $a_1 = 3$, $a_2 = 3$ and we obtain

$$v(S) = 3r + 3r^2.$$

Allocation rule

We propose here the procedure of allocation the general gain $v(N)$ to each player $i \in N$.
Stage 1. Two direct connected players obtain r. Individually, they don't receive nothing. So, each of them hopes to receive at least $r/2$. If player i has some direct connections then she receives the value $r/2$ times the number of paths of length 1 wnich contain the node i.
Stage 2. Three connected players obtain r^2, so each of them must receive $r^2/3$.
Arguing the same way we obtain the allocation rule of the following form:

$$Y_i(v,g) = \frac{A_1^i}{2} r + \frac{A_2^i}{3} r^2 + \cdots + \frac{A_L^i}{L+1} r^L = \sum_{k=1}^{L} \frac{A_k^i}{k+1} r^k, \qquad (3)$$

where A_k^i is the number of all paths of length k which contain the node i.

Example

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Let us calculate the payoff to player 2 in example 1. Mark all paths which contain the node 2. The paths of length 1 are: $\{1,2\}$, $\{2,4\}$, $\{2,5\}$, hence $A_1^2 = 3$. The paths of length 2 are: $\{1,2,4\}$, $\{1,2,5\}$, $\{4,2,5\}$, $\{2,1,3\}$, so $A_2^2 = 4$. The paths of length 3: $\{3,1,2,4\}$, $\{3,1,2,5\}$, $\{2,1,3,6\}$, $A_3^2 = 3$. The paths of length 4 are: $\{4,2,1,3,6\}$, $\{5,2,1,3,6\}$, $A_4^2 = 2$. Consequently,

$$Y_2 = \frac{3}{2}r + \frac{4}{3}r^2 + \frac{3}{4}r^3 + \frac{2}{5}r^4.$$

Generating function for computing the Myerson value

Consider the tree $g_p = (N, E)$ with the root in the node p.
Introduce the generating function

$$\varphi_p(x) = \sum_{k=1}^{L} \alpha_k^p x^k$$

where α_k^p is the number of paths which consist on k nodes (length $k-1$) and contain the node p.
To find this value we use modified algorithm proposed by Jamison for computing the generating function for the number of sub-trees of a tree g which contain k nodes of the tree g.

Calculate the generating function via recurrence relations.
First, we determine in final nodes q of the tree g_p

$$\varphi_q(x) = x.$$

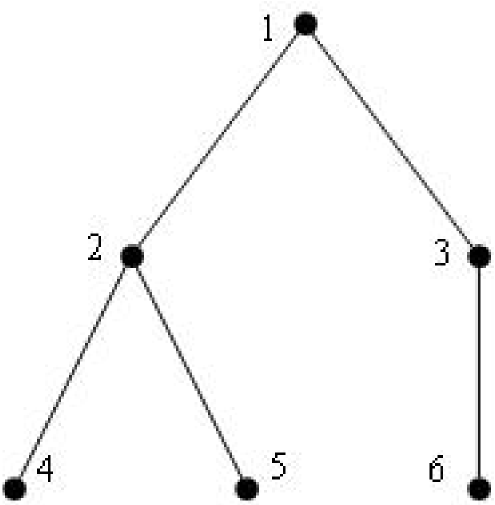
Denote l the maximal length $\{p, \ldots, q\}$. Consider the nodes q such that the length of path $\{p, \ldots, q\}$ is equal to $l-1$. If it is not the root p then assume

$$\varphi_q(x) = x\left(1 + \sum_{i=1}^{d} \varphi_{q_i}(x)\right)$$

where the sum is calculated in all descendants $q_i, i = 1, \ldots, d$ of the node q. Continue the process until $l = 2$. For $l = 2$ the generating function is determined for all descendants of p.

$$\varphi_p(x) = x\left(1 + \sum_{i=1}^{d} \varphi_{q_i}(x) + \sum_{i \neq j} \varphi_{q_i}(x)\varphi_{q_j}(x)\right)$$

Example



Example

For the tree in example 1 for first player we obtain
$\varphi_4(x) = \varphi_5(x) = \varphi_6(x) = x$;
$\varphi_2(x) = x(1 + \varphi_4(x) + \varphi_5(x)) = x(1 + 2x)$;
$\varphi_3(x) = x(1 + \varphi_6(x)) = x(1 + x)$;
$\varphi_1(x) = x(1 + \varphi_2(x) + \varphi_3(x) + \varphi_2(x)\varphi_3(x)) =$
$x + 2x^2 + 4x^3 + 3x^4 + 2x^5$.
It yields
$A_1^1 = \alpha_2^1 = 2$;
$A_2^1 = \alpha_3^1 = 4$;
$A_3^1 = \alpha_4^1 = 3$;
$A_4^1 = \alpha_5^1 = 2$.

Cooperative game and the Myerson value

Myerson allocation rule

$$Y(v,g) = (Y_1(v,g), \ldots, Y_n(v,g)),$$

is uniquely determined by the following axioms:
A1. If S is a component of g then the members of the coalition S ought to allocate to themselves the total value $v(S)$ available to them, i.e $\forall S \in N|g$

$$\sum_{i \in S} Y_i(v,g) = v\,(S)\,. \tag{4}$$

A2. $\forall g,\ \forall ij \in g$ both players i and j obtain equal payoffs after adding or deleting a link ij,

$$Y_i\,(v,g) - Y_i\,(v,g-ij) = Y_j\,(v,g) - Y_j\,(v,g-ij)\,. \tag{5}$$

Myerson value for unweighted network

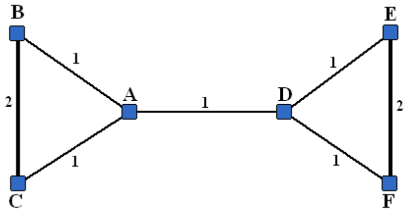
Allocation rule (6) satisfies A1-A2.

$$Y_i\,(w, g) = \frac{\sigma_1(i)}{2} r + \frac{\sigma_2(i)}{3} r^2 + \cdots + \frac{\sigma_L(i)}{L+1} r^L = \sum_{k=1}^{L} \frac{\sigma_k(i)}{k+1} r^k, \quad (6)$$

where $\sigma_k(i)$ is a number of the paths of the length k which include i.

Theorem. This allocation rule is the Myerson value.

Example. Network of six nodes



.: Network of six nodes.

For the network $N = \{A, B, C, D, E, F\}$ we find $L = 3$, $a_1 = 9$, $a_2 = 4$, $a_3 = 4$. Consequently, the value of grand-coalition is

$$v(N) = 9r + 4r^2 + 4r^3.$$

For coalition $S = \{A, B, C, D\}$ we have $L = 2$, $a_1 = 5$, $a_2 = 2$ and we obtain

$$v(S) = 5r + 2r^2.$$

Example. Myerson value

Let us calculate the Myerson value for player A in Example 1 using the allocation rule (6). Mark all paths which contain node A.
The paths of length 1 are: {A,B}, {A,C}, {A,D}, hence $a_1^A = 3$.
The paths of length 2 are: {B,A,D}, {C,A,D}, {A,D,E}, {A,D,F}, so $a_2^A = 4$.
The paths of length 3: {B,A,D,E}, {B,A,D,F}, {C,A,D,E}, {C,A,D,F}, so $a_3^A = 4$.
Consequently,

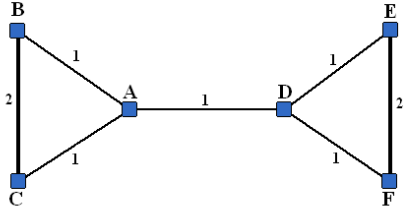
$$Y_A = \frac{3}{2}r + \frac{4}{3}r^2 + r^3.$$

Network partitioning. Nash stability

The following algorithm for network partitioning based on the Myerson value:
Start with any partition of the network $N = \{S_1, \ldots, S_K\}$.
Consider a coalition S_l and a player $i \in S_l$. In cooperative game presented by the graph $g|S_l$ we find the Myerson value for player i, $Y_i(g|S_l)$. That is reward of player i in coalition S_l.
Suppose that player i decides to join the coalition S_k. In the new cooperative game with partial cooperation presented by the graph $g|S_k \cup i$ we find the Myerson value $Y_i(g|S_k \cup i)$.
So, if for the player $i \in S_l$: $Y_i(g|S_l) \geq Y_i(g|S_k \cup i)$ then player i has no incentive to join to new coalition S_k, otherwise the player changes the coalition.
Defonition. The partition $N = \{S_1, \ldots, S_K\}$ is the Nash stable if for any player there is no incentive to move from her coalition.

Nash stability

The network



Natural way of partition here is $\{S_1 = (A, B, C), S_2 = (D, E, F)\}$. Let us determine under which condition this structure will present the stable partition.

For coalition S_1 the payoff $v(S_1) = 4r$. The payoff of player A is $Y_A(g|S_1) = r$. Imagine that player A decides to join the coalition S_2.

Nash stability

Coalition $S_2 \cup A$ has payoff $v(S_2 \cup A) = 5r + 2r^2$.
The imputation in this coalition is
$Y_A(g|S_2 \cup A) = r/2 + 2r^2/3$, $Y_D(g|S_2 \cup A) = 3r/2 + 2r^2/3$, $Y_E(g|S_2 \cup A) = Y_F(g|S_2 \cup A) = 3r/2 + r^3/3$.
We see that for player A it is profitable to join this new coalition if $r/2 + 2r^2/3 > r$, or $r > 3/4$. Otherwise, the coalition structure is Nash stable.
Thus, for the network in Fig. 1 the Myerson value approach will give the partition $\{S_1 = (A, B, C), S_2 = (D, E, F)\}$ if $r < 3/4$ and, otherwise, it leads to the grand coalition.

Hedonic coalition game approach

Game with hedonic coalitions, where a player's payoff is completely determined by the identity of other members of his coalition [Bogomolnaia, Jackson].

Assume that the set of players $N = \{1, \ldots, n\}$ is divided into K coalitions: $\Pi = \{S_1, \ldots, S_K\}$. Let $S_\Pi(i)$ denote the coalition $S_k \in \Pi$ such that $i \in S_k$. A player i preferences are represented by a complete, reflexive and transitive binary relation $\succeq_i$ over the set $\{S \subset N : i \in S\}$. The preferences are additively separable if there exists a value function $v_i : N \to \mathbb{R}$ such that $v_i(i) = 0$ and

$$S_1 \succeq_i S_2 \Leftrightarrow \sum_{j \in S_1} v_i(j) \geq \sum_{j \in S_2} v_i(j).$$

Hedonic coalition game approach

The preferences $\{v_i, i \in N\}$ are symmetric $v_i(j) = v_j(i) = v_{ij} = v_{ji}$, $\forall i, j \in N$.

Definition. The network partition Π is *Nash stable*, if $S_\Pi(i) \succeq_i S_k \cup \{i\}$ for all $i \in N, S_k \in \Pi \cup \{\emptyset\}$. In the Nash-stable partition, there is no player who wants to leave her coalition.

A potential of a coalition partition $\Pi = \{S_1, \dots, S_K\}$ is

$$P(\Pi) = \sum_{k=1}^{K} P(S_k) = \sum_{k=1}^{K} \sum_{i,j \in S_k} v_{ij}. \quad (7)$$

Network partitioning

Start with any partition of the network $N = \{S_1, \ldots, S_K\}$. Choose any player i and any coalition S_k different from $S_\Pi(i)$. If

$$S_k \cup \{i\} \succeq_i S_\Pi(i)$$

assign node i to the coalition S_k; otherwise, keep the partition unchanged and choose another pair of node-coalition, etc.
Since the game has the potential (7), the above algorithm is guaranteed to converge in a finite number of steps.
Proposition. *If players' preferences are additively separable and symmetric ($v_{ii} = 0, v_{ij} = v_{ji}$ for all $i, j \in N$), then the coalition partition Π giving a local maximum of the potential $P(\Pi)$ is the Nash-stable partition.*

Network partitioning

Define a value function v with a parameter $\alpha \in [0,1]$ is as follows:

$$v_{ij} = \begin{cases} 1-\alpha, & (i,j) \in E, \\ -\alpha, & (i,j) \notin E, \\ 0, & i = j. \end{cases} \tag{8}$$

For any subgraph $(S, E|S)$, $S \subseteq N$, denote $n(S)$ as the number of nodes in S, and $m(S)$ as the number of edges in S. Then, for the value function (8), the potential (7) takes the form

$$P(\Pi) = \sum_{k=1}^{K} \left(m(S_k) - \frac{n(S_k)(n(S_k)-1)\alpha}{2} \right). \tag{9}$$

Decomposition of network

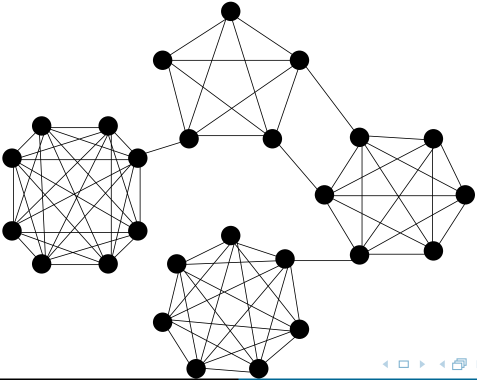
Introduce a special decomposition of the network into the cliques.
At first, let us find a maximum clique S_1 in the network G (a maximum clique of a graph, is a clique, such that there is no clique with more vertices).
Skip all vertices of S_1 from G and consider the new network G'.
Let us find a maximum clique S_2 in the network G' and continue this procedure until we derive the partition $\{S_1, ..., S_K\}$ of the network G into cliques.
Call this partition sequential decomposition of the network into maximum cliques.

Network partitioning

Proposition 2. *If* $\alpha = 0$*, the grand coalition partition* $\Pi_N = \{N\}$ *gives the maximum of the potential* (8)*. Whereas if* $\alpha \to 1$*, the maximum of* (9) *corresponds to a network sequential decomposition into maximum cliques.*

Network partitioning

Example. Consider graph $G = G_1 \cup G_2 \cup G_3 \cup G_4$ Which consists of $n = 26$ nodes and $m = 78$ edges. This graph includes 4 full connected subgraphes: $(G_1, 8, 28)$ with 8 vertices connected by 28 links, $(G_2, 5, 10)$, $(G_3, 6, 15)$ and $(G_4, 7, 21)$. Subgraph G_1 is connected with G_2 by 1 edge, G_2 with G_3 by 2 edges, and G_3 with G_4 by 1 edge.



Network partitioning

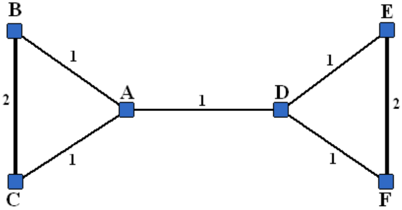
Firstly, find the potentials (9) for large-scale decompositions of G for any parameter $\alpha \in [0, 1]$. It is easy to check, that
$P(G) = 78 - 325\alpha$, $P(\{G_1, G_2 \cup G_3 \cup G_4\}) = 77 - 181\alpha$,
$P(\{G_1, G_2 \cup G_3, G_4\}) = 76 - 104\alpha$,
$P(\{G_1, G_2, G_3, G_4\}) = 74 - 74\alpha$.
Other coalition partitions give smaller potentials:
$P(\{G_1 \cup G_2, G_3 \cup G_4\}) = 76 - 156\alpha < 76 - 104\alpha$,
$P(\{G_1 \cup G_2 \cup G_3, G_4\}) = 77 - 192\alpha < 77 - 181\alpha$,
$P(\{G_1, G_2, G_3 \cup G_4\}) = 75 - 116\alpha < 76 - 104\alpha$,
$P(\{G_1 \cup G_2, G_3, G_4\}) = 75 - 114\alpha < 76 - 104\alpha$.

Network partitioning

Nash-stable coalition partitions in Example 2.

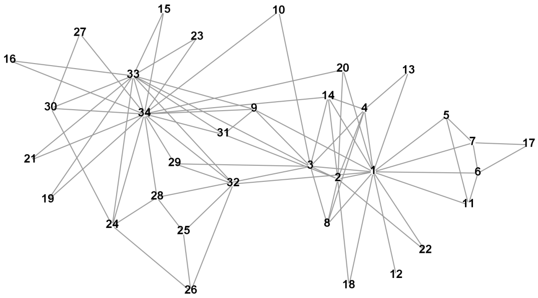
α	coalition partition	potential
$[0, 1/144]$	$G_1 \cup G_2 \cup G_3 \cup G_4$	$78 - 325\alpha$
$[1/144, 1/77]$	$G_1, G_2 \cup G_3 \cup G_4$	$77 - 181\alpha$
$[1/77, 1/15]$	$G_1, G_2 \cup G_3, G_4$	$76 - 104\alpha$
$[1/15, 1]$	G_1, G_2, G_3, G_4	$74 - 74\alpha$

Network partitioning



For the unweighted version of the network example presented in Fig, there are only two stable partitions: $\Pi = N$ for small values of $\alpha \leq 1/9$ and $\Pi = \{\{A, B, C\}, \{D, E, F\}\}$ for $\alpha > 1/9$.

Network "Zachary karate club"



.: Zachary karate club network.

Zachary [1977] observed 34 members of a karate club over a period of two years. Due to a disagreement developed between the administrator of the club and the club instructor there appear two new clubs associated with the instructor (node 1) and administrator (node 34) of sizes 15 and 19, respectively.

Network "Zachary karate club"

Girvan, Newman [2002] divide the network into two groups of roughly equal size using the hierarchical clustering tree. They show that this split corresponds almost perfectly with the actual division of the club members following the break-up. Only one node, node 3, is classified incorrectly.

Hedonic game approach

Let us now apply the hedonic game approach to the karate club network. We start from the final partition $N = \{S_{15}, S_{19}\}$, which was obtained in Girvan, Newman.
We calculate the potential for grand-coalition $P(N) = 78 - 561\alpha$ and for partition $P(S_{15}, S_{19}) = 68 - 276\alpha$.
If $\alpha < 2/57$, $P(N)$ is larger than $P(S_{15}, S_{19})$, so partition $\{S_{15}, S_{19}\}$ is not Nash-stable.
For $\alpha = 2/57$ the potential increases if the node 3 moves from S_{19} to S_{15}. For the new partition $P(S_{16}, S_{18}) = 68 - 273\alpha$.
For $\alpha = 5/144$ the potential increases if the node 10 moves to S_{16}.
Thus, for $\alpha \geq 10/289$ the Nash-stable partition is

$$S_{17} = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 17, 18, 20, 22\} \cup \{N \backslash S_{17}\}.$$

Notice that in this new partition the node 3 belongs to the "right" coalition.

Modularity

Define a value function via the configuration random graph model. The configuration random graph model can be viewed as a null model for a network with no community structure. Namely, the following value function can be considered:

$$v_{ij} = \beta_{ij}\left(A_{ij} - \gamma\frac{d_i d_j}{2m}\right), \tag{10}$$

where A_{ij} is a number of links between nodes i and j, d_i and d_j are the degrees of the nodes i and j, respectively, $m = \frac{1}{2}\sum_{l\in N} d_l$ is the total number of links in the network, and $\beta_{ij} = \beta_{ji}$ and γ are some parameters.

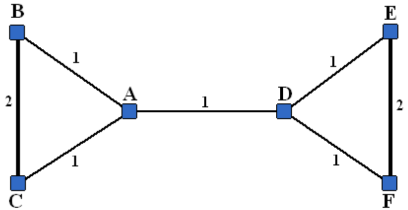
Modularity

Thus, we have now a game-theoretic interpretation of the modularity function. Namely, the coalition partition $\Pi = \{S_1, \ldots, S_K\}$ which maximises the modularity

$$P(\Pi) = \sum_{k=1}^{K} \sum_{i,j \in S_k, i \neq j} \left(A_{ij} - \frac{d_i d_j}{2m} \right) \tag{11}$$

gives the Nash-stable partition of the network in the Hedonic game with the value function defined by (10), where $\gamma = 1$ and $\beta_{ij} = \beta$.

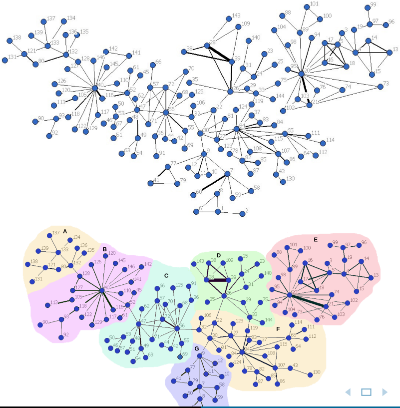
Modularity



For the network presented in Figure we calculate $P(N) = 3/2, P(\{B, C\} \cup \{A, D\} \cup \{E, F\}) = P(\{A, B, C, D\} \cup \{E, F\}) = 7/2$ and $P(\{A, B, C\} \cup \{D, E, F\}) = 5$. Thus, according to the value function (10) with $\gamma = 1$ and $\beta_{ij} = \beta$ (modularity value function), $\Pi = \{\{A, B, C\}, \{D, E, F\}\}$ is the unique Nash-stable coalition.

Math-net.ru partitioning

For simplicity we delete all links with weight which is smaller than seven. The result is presented in Figure. It follows that the nodes 40, 34, 56 and 20 are the centers of 'local' stars and, consequently, must have a high centrality. Notice that node 32 also must have high centrality because it is connecting two separate components.



Social networks $\Gamma =< N, G, R >$, where
$N = \{1, ..., n\}$ - players,
G -graph of network,
$R = \{r_{ij}\}$ -reputation matrix, $r_{ij} > 0$ - rate of reputation $i \to j,\ i, j \in N$.
R-stochastic matrix.
Let $x(t) = (x_1(t), ..., x_n(t))$ - vector of decisions. $x(0)$ - initial vector.
$x(t) = Rx(t-1), t = 1, 2, ...$ - conversations
or
$$x_i(t) = \sum_{j \in N} r_{ij} x_j(t-1), t = 1, 2, ...$$

Social networks

$\Rightarrow x(t) = R^t x(0)$
As $t \to \infty$ then $x(t) \to x = (x_1, ..., x_n)$ - vector final decisions.
$x = R^\infty x(0)$ where $R^\infty = \lim R(t)$.
R-stochastic matrix $\Rightarrow R^\infty = (r, r, ..., r)'$.
Thus, $x = (X, X, ..., X)$ - final decisions of all players are equal.
$r = (r_1, ..., r_n)$ - power indexes.

$$X = \sum_{i \in N} r_i x_i(0)$$

is consensus.

Information Fighting

Two Bosses I and II.
I controls coalition $S_1 \subset N$, recommended decision d_1,
II controls $S_2 \subset N$, recommended decision d_2,
$S_3 = N - S_1 - S_2$-neutral players.
So, final decision is

$$X = \rho_1 d_1 + \rho_2 d_2 + \sum_{j \in S_3} r_j x_j(0),$$

where

$$\rho_1 = \sum_{j \in S_1} r_j, \ \rho_2 = \sum_{j \in S_2} r_j.$$

Information fighting game is $< I, II, D, D, H_1(d_1, d_2), H_2(d_1, d_2) >$,
where $H_i(d_1, d_2) = H_i(X, d_i), i = 1, 2$.

Information Fighting

Let $H_1 = X - 2X^2 - c_1 d_1^2,\ H_2 - X - 3X^2 - c_2 d_2^2$
or

$$H_1(d_1, d_2) = (d_1\rho_1 + d_2\rho_2) - 2(d_1\rho_1 + d_2\rho_2)^2 - c_1 d_1^2$$

$$H_2(d_1, d_2) = (d_1\rho_1 + d_2\rho_2) - 3(d_1\rho_1 + d_2\rho_2)^2 - c_2 d_2^2.$$

First order condition

$$\frac{\partial H_1}{\partial d_1} = \rho_1 - 4(d_1\rho_1 + d_2\rho_2) - 2c_1 d_1 = 0.$$

$$\frac{\partial H_2}{\partial d_2} = \rho_2 - 6(d_1\rho_1 + d_2\rho_2) - 2c_2 d_2 = 0.$$

Information Fighting

Nash equilibrium is

$$d_1^* = \frac{\rho_1(c_2 + \rho_2^2)}{2c_1c_2 + 6c_1\rho_2^2 + 4c_2\rho_1^2}$$

$$d_2^* = \frac{\rho_2(c_1 - \rho_1^2)}{2c_1c_2 + 6c_1\rho_2^2 + 4c_2\rho_1^2}.$$

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