NETWORKING GAMES AND APPLICATIONS

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Content

Networking game

Game=<Players, Strategies, Payoffs>
Players: Programs, Packages, Mobile Phones, Nodes of Networks, etc.
Strategies: balancing, routing, level of signals, number of links, etc.
Payoffs: cost, delay, working time, profit, reward, etc.

Formal definition:
$\Gamma =< N = 1, 2, ..., n, \{X_i\}_{i \in N}, \{H_i(x_1, ..., x_n)\}_{i \in N} >$
$x = (x_1, ..., x_n)$ - profile of strategies.
Objective of player $i \in N$ $H_i(x_1, ..., x_n) \to \max_{x_i}$ or $\min_{x_i}$

Nash Equilibrium and Strong Nash Equilibrium

We use notation $x = (x_i, x_{-i})$ for $x = (x_1, ..., x_i, ..., x_n)$ and for coalition $S = \{i_1, ..., i_k\}$ (a subset of N) let $x = (x_S, x_{-S})$ for $x = (x_1, ..., x_{i_1}, ..., x_{i_k}, ..., x_n)$.

Definition 1. Profile $x^* = (x_1^*, ..., x_n^*)$ is Nash equilibrium if for any $i \in N$

$$H_i(x_i, x_{-i}^*) \leq H_i(x^*), \quad \forall x_i.$$

Definition 2. Profile $x^* = (x_1^*, ..., x_n^*)$ is Strong Nash equilibrium if for any coalition $S \subset N$ and any profile x_S it exists a player $i \in S$ for whom

$$H_i(x_S, x_{-S}^*) \leq H_i(x^*).$$

Denote $H(x_1,...,x_n) = \sum_{i \in N} H_i(x_1,...,x_n)$. Profile x_{opt} which maximizes $H(x_1,...,x_n)$ we call the **cooperative solution**.

Definition 3. Nash equilibrium x_{WNE} is worst case Nash equilibrium if for any NE x

$$H(x_{WNE}) \leq H(x).$$

Definition 4. Let x_{WNE} is the worst case Nash equilibrium and x_{opt} cooperative solution. Then the ratio

$$PoA = \frac{H(x_{WNE})}{H(x_{opt})}$$

we call Price of Anarchy (Papadimitriou [1999]).

Cooperation and Competition

By analogy for costs $C_i(x_1, ..., x_n), i \in N$ denote **Social Cost** $C(x_1, ..., x_n) = \sum_{i \in N} C_i(x_1, ..., x_n)$. Profile x_{opt} which minimizes $C(x_1, ..., x_n)$ we call the **cooperative solution**.

Definition 5. Nash equilibrium x_{WNE} is worst case Nash equilibrium if for any NE x

$$C(x) \leq C(x_{WNE}).$$

Definition 6. Let x_{WNE} is the worst case Nash equilibrium and x_{opt} cooperative solution. Then the ratio

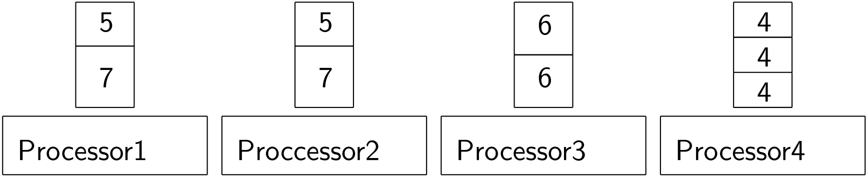
$$PoA = \frac{C(x_{WNE})}{C(x_{opt})}$$

we call Price of Anarchy.

Load Balancing

$n = 4$ processors, the jobs $w = (7, 7, 6, 6, 5, 5, 4, 4, 4)$.

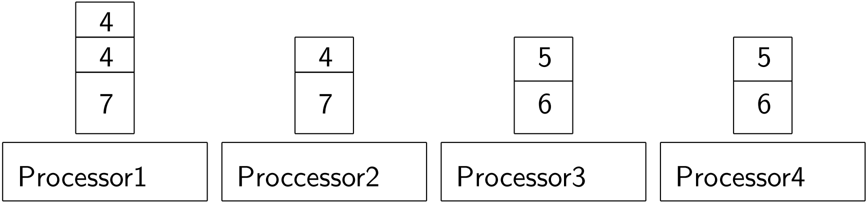
Optimal Load



Optimal makespan (OPT) 12.

Load Balancing. Nash Equilibrium

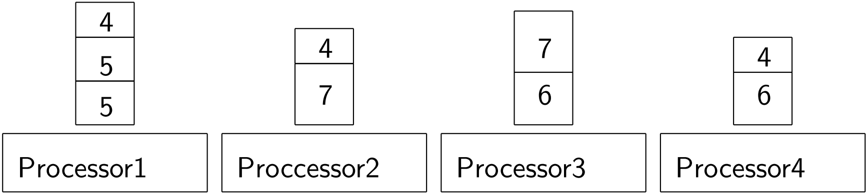
$n = 4$ processors, the jobs $w = (7, 7, 6, 6, 5, 5, 4, 4, 4)$.
Nash Equilibrium



Maximal load (Social Cost) 15.
Price of Anarchy 15/12=1.25
Coalition (7, 4, 4, 5, 5) improves payoffs

Load Balancing. Strong Nash Equilibrium

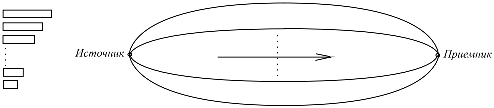
$n = 4$ processors, the jobs $w = (7, 7, 6, 6, 5, 5, 4, 4, 4)$.
Strong Nash Equilibrium



Maximal load (Social Cost) 14.
Strong Price of Anarchy 14/12=1.166

Routing problem

Network with m parallel links.



n users send the traffic of size w_i in one of links $i = 1, \dots, n$. Each link $l = 1, \dots, m$ has capacity c_l. Assume that the delay for traffic w in the link with c is equal to w/c.
A user tries to minimize the delay.
$i \to l_i$: $L = (l_1, \dots, l_n)$ is profile of strategies.
Mixed strategy is $p_i = (p_i^1, \dots, p_i^m)$, where p_i^l – probability that i uses l.
Matrix P is profile of all strategies.

Routing problem

Pure strategies: The delay of player i in link l_i is

$$\lambda_i = \frac{\sum_{k:l_k=l_i} w_k}{c_{l_i}}.$$

Def. 1. Profile $(l_1, \ldots, l_n)$ is NE if for any user i
$\lambda_i = \min_{j=1,\ldots,m} \frac{w_i + \sum_{k \neq i: l_k = j} w_k}{c_j}$.

Mixed strategies: the expected delay of user i using link l is
$\lambda_i^l = \frac{w_i + \sum\limits_{k=1, k\neq i}^{n} p_k^l w_k}{c_l}$.
Minimal expected delay is $\lambda_i = \min_{l=1,\ldots,m} \lambda_i^l$.

Def. 2. Profile P is NE if for any user i and any link is satisfied
$\lambda_i^l = \lambda_i$ for $p_i^l > 0$, and $\lambda_i^l > \lambda_i$, for $p_i^l = 0$.

Fully mixed equilibrium

Def. 3. P is fully mixed NE if each user chooses the link in NE with $p_i^l > 0$.

$$\lambda_i^l = \frac{w_i + \sum\limits_{k=1, k \neq i}^{n} p_k^l w_k}{c_l} = \lambda_i, \forall i, l.$$

Social costs

Social costs $SC(w, c, L)$ for pure strategies are:

Linear costs $LSC(w, c, L) = \sum_{l=1}^{m} \frac{\sum_{k:l_k=l} w_k}{c_l}$;

Quadratic costs $QSC(w, c, L) = \sum_{l=1}^{m} \frac{\left(\sum_{k:l_k=l} w_k\right)^2}{c_l}$;

Maximal costs $MSC(w, c, L) = \max_{l=1,\dots,m} \frac{\sum_{k:l_k=l} w_k}{c_l}$.

Social costs and Price of Anarchy

Def. 4. For mixed profile P social costs are

$$SC(w,c,P) = E\left(SC(w,c,L)\right) = \sum_{L=(l_1,\ldots,l_n)} \left(\prod_{k=1}^{n} p_k^{l_k} \cdot SC(w,c,L)\right).$$

Optimal social costs $opt = \min_P SC(w,P)$.

Def. 5. Price of anarchy is the ratio of the social costs in worst case of NE to the optimal costs

$$PA = \sup_{P-\text{NE}} \frac{SC(w,P)}{opt}.$$

$PA \geq 1$.

Worst Case Nash Equilibrium

Let $n = 5$ players , $m = 3$ links, $w = (20, 10, 10, 10, 5)$, $c = (20, 10, 8)$.

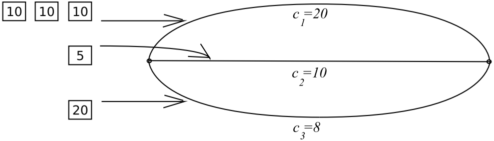


Fig. 2. Worst case of NE with delay 2.5

There are few NE. $\{(10, 10, 10) \to 20, 5 \to 10, 20 \to 8)\}$.
For this profile SC are maximal

$$MSC(w; c; (10, 10, 10) \to 20, 5 \to 10, 20 \to 8) = 2.5.$$

This NE is worst case NE. The maximum of SC is achieved in $(20, 10) \to 20, (10, 5) \to 10, 10 \to 8$ and equal to 1.5.
$PoA = \frac{2.5}{1.5} = 5/3$.

Braess paradox

Delete the link 8, then in worst case the social costs are

$$MSC(w; c; (20, 10, 10) \to 20, (10, 5) \to 10) = 2.$$

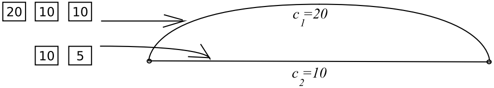


Fig. 3. The delay is decreasing if we delete a link

$PoA = \frac{2}{1.5} = 4/3.$

Examples

Example 2. $n = 4$, $m = 3$, $w = (15, 5, 4, 3)$, $c = (15, 10, 8)$. The social costs in worst case NE are

$$MSC(w; c; (5, 4) \to 15, 15 \to 10, 3 \to 8) = 1.5.$$

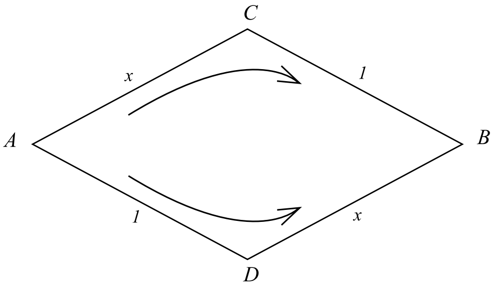
Optimal load $15 \to 15, (5, 3) \to 10, 4 \to 8$, makespan is 1.
If we delete the link 10 then the worst case NE is $(15, 5) \to 15, (4, 3) \to 8$ with SC =1.333. Global optimum and the best NE are achieved for $(15, 3) \to 15, (5, 4) \to 8$, and SC= 1.2.
Example 3. $n = 4$, $m = 3$, $w = (15, 8, 4, 3)$, $c = (15, 8, 3)$. SC for worst case NE are

$$MSC(w; c; (8, 4, 3) \to 15, 15 \to 8) = 1.875.$$

Optimal load $(15, 4) \to 15, 8 \to 8, 4 \to 3$, makespan is 1.2666.
If we delete link 8 the worst case NE is $(15, 8, 4) \to 15, 3 \to 3$ with SC= 1.8. Global optimum and the best NE are $(15, 8, 3) \to 15, 4 \to 3$, and SC= 1.733.

NE in pure strategies. Braess paradox

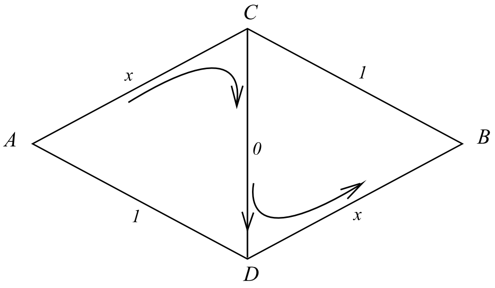
Example. Braess paradox. Consider the system of roads. Suppose 60 cars move from A to B. The delay in the links (C, B) and (A, D) doesn't depended of number of cars and equal to 1 hour, in the links (A, C) and (D, B) is proportional the number of cars (in minutes). We find that NE is the equal distribution of cars in the links (A, C, B) and (A, D, B). That is 30 cars in each link. So, the delay of each player is 1.5 hours.



Users are distributed uniformly in both links

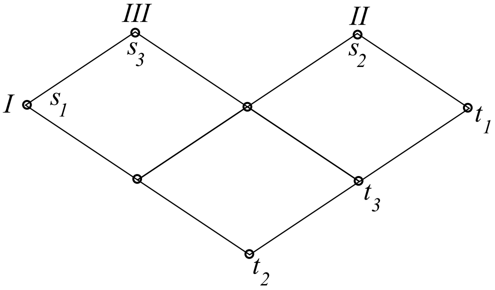
NE in pure strategies. Braess paradox

Suppose that we connect C and D bu new road with delay 0. Then for a car which drives in the link (A, D, B) is more profitable to drive in (A, C, D, B). The same for cars in (A, C, B), more profitable to drive in (A, C, D, B). Thus, NE is (A, C, D, B). But the delay for each player now is 2 hours.



Price of anarchy. General network

The users $N = (1, 2, ..., n)$ send traffic in network $G = (V, E)$, V nodes and E edges. For each user i it is determined set Z_i of admissable paths from s_i to t_i by G. We suppose that all users send the unit traffic.



For each link $e \in E$ it is determined capacity $c_e > 0$. Each user tries to minimize the delay sending traffic from s to t.

Strategies. Linear latency

The strategy $R_i \in Z_i$, is a path. Then $R = (R_1, \ldots, R_n)$ is a profile of strategies. For profile R we write $(R_{-i}, R'_i) = (R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_n)$. It means that user i changes the strategy from R_i to R'_i, and other players use the same strategies.

For each link let $n_e(R)$ is the number of players using link e in profile R. The delay depends on the load of the used links. Let consider the latency in linear form

$$f_e(k) = a_e k + b_e,$$

where a_e and b_e non-negative constants. For simplicity let $f_e(k) = k$.

The costs

Each user minimizes the sum of latencies in all links. The personal costs of user i is

$$c_i(R) = \sum_{e \in R_i} f_e(n_e(R)) = \sum_{e \in R_i} n_e(R).$$

NE is the profile where nobody is interested to change his strategies.

Definition. Profile R is NE if for each user $i \in N$ we have $c_i(R) \leq c_i(R_{-i}, R_i')$.

Linear social costs

Social costs here are linear

$$SC(R) = \sum_{i=1}^{n} c_i(R) = \sum_{i=1}^{n} \sum_{e \in R_i} n_e(R) = \sum_{e \in E} n_e^2(R).$$

Minimal SC are *opt*. Let find the ratio

$$PoA = \sup_{R - \mathrm{NE}} \frac{SC(R)}{opt}.$$

Price of anarchy. General network

Theorem. Price of anarchy is equal to 5/2.
Proof. Let R^* is NE and R is any profile (in particular, optimal). For NE R^* the personal cost of user i if he switches for the strategy R_i will increased

$$c_i(R^*) = \sum_{e \in R_i^*} n_e(R^*) \le \sum_{e \in R_i} n_e(R^*_{-i}, R_i).$$

If player i switches then the load of each link can increased at most for 1, hence,

$$c_i(R^*) \le \sum_{e \in R_i} (n_e(R^*) + 1).$$

Summarizing these inequalities in i we obtain

$$SC(R^*) = \sum_{i=1}^{n} c_i(R^*) \le \sum_{i=1}^{n} \sum_{e \in R_i} (n_e(R^*)+1) = \sum_{e \in E} n_e(R)(n_e(R^*)+1).$$

Price of anarchy. General network

Lemma. For any non-negative integer numbers α, β is takes place

$$\beta(\alpha+1) \leq \frac{1}{3}\alpha^2 + \frac{5}{3}\beta^2.$$

From lemma

$$SC(R^*) \leq \frac{1}{3}\sum_{e \in E} n_e^2(R^*) + \frac{5}{3}\sum_{e \in E} n_e^2(R) = \frac{1}{3}SC(R^*) + \frac{5}{3}SC(R),$$

hence

$$SC(R^*) \leq \frac{5}{2}SC(R)$$

for any profile R. It yields $PoA \leq 5/2$.

Potential game

$\Gamma =< N = 1, 2, ..., n, \{X_i\}_{i \in N}, \{H_i(x_1, ..., x_n)\}_{i \in N} >$
$x = (x_1, ..., x_n)$ - profile of strategies.
Objective of player $i \in N$ $H_i(x_1, ..., x_n) \to \max\limits_{x_i}$ or $\min\limits_{x_i}$
We use notation $x = (x_i, x_{-i})$ for $x = (x_1, ..., x_i, ..., x_n)$.
Definition 1. Profile $x^* = (x_1^*, ..., x_n^*)$ is Nash equilibrium if for any $i \in N$

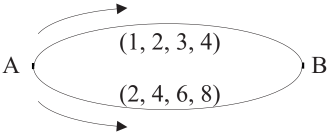
$$H_i(x_i, x_{-i}^*) \leq H_i(x^*), \quad \forall x_i.$$

A normal-form n-player game $\Gamma =< N, \{X_i\}_{i\in N}, \{H_i\}_{i\in N} >$.

Suppose that there exists a certain function $P : \prod_{i=1}^{n} X_i \to R$ such that for any $i \in N$ we have the inequality

$$H_i(x_{-i}, x_i') - H_i(x_{-i}, x_i) = P(x_{-i}, x_i') - P(x_{-i}, x_i)$$

for arbitrary $x_{-i} \in \prod_{j\neq i} X_j$ and any strategies $x_i, x_i' \in X_i$. Then Γ is potential game and P is a potential function.



Traffic jamming. Suppose that players *I* and *II*, each possessing two packages, have to deliver it from point A to point B.

These points communicate through two links. Numbers on the figure indicate the journey time on each link depending on the number of moving packages.

Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-8,-8) & (-6,-5) & (-4,-8) \\ (-5,-6) & (-6,-6) & (-7,-12) \\ (-8,-4) & (-12,-7) & (-16,-16) \end{pmatrix} \end{array}.$$

Potential games

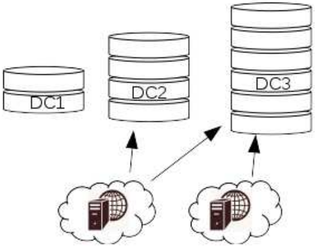
Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-8,-8) & (-6,-5) & (-4,-8) \\ (-5,-6) & (-6,-6) & (-7,-12) \\ (-8,-4) & (-12,-7) & (-16,-16) \end{pmatrix} \end{array}.$$

The game possesses the potential

$$P = \begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} 13 & 16 & 13 \\ 16 & 16 & 10 \\ 13 & 10 & 0 \end{pmatrix} \end{array}$$

Potential games



Choice of data centers. Assume each of two cloud operators may conclude a contract to utilize the capacity resources of one or two of three data centers available. The resources of data centers 1, 2, and 3 are 2, 4, and 6, respectively. If both operators choose the same data center, they equally share its resources. The payoff of each player is the sum of the obtained resources at each segment minus the rent cost of the resources provided by a data center (let this cost be 1).

Potential games

Payoff matrix:

$$
\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array}
\begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\
\begin{pmatrix}
(0,0) & (1,3) & (1,5) & (0,3) & (0,5) & (1,8) \\
(3,1) & (1,1) & (3,5) & (1,2) & (3,6) & (1,6) \\
(5,1) & (5,3) & (2,2) & (5,4) & (2,3) & (2,5) \\
(3,0) & (2,1) & (4,5) & (1,1) & (3,5) & (2,6) \\
(5,0) & (6,3) & (3,2) & (5,3) & (2,2) & (3,5) \\
(8,1) & (6,1) & (5,2) & (6,2) & (5,3) & (3,3)
\end{pmatrix} \end{array}.
$$

Potential:

$$
P = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array}
\begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\
\begin{pmatrix}
1 & 4 & 6 & 4 & 6 & 9 \\
4 & 4 & 8 & 5 & 9 & 9 \\
6 & 8 & 7 & 9 & 8 & 10 \\
4 & 5 & 9 & 5 & 9 & 10 \\
6 & 9 & 8 & 9 & 8 & 11 \\
9 & 9 & 10 & 10 & 11 & 11
\end{pmatrix} \end{array}.
$$

Potential games

Theorem. *Let an n-player game* $\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >$ *have a potential* P*. Then a Nash equilibrium in the game* Γ *represents a Nash equilibrium in the game* $\Gamma' =< N, \{X_i\}_{i \in N}, P >$*, and vice versa. Furthermore, the game* Γ *admits at least one pure strategy equilibrium.*

Proof. The first assertion follows from the definition of a potential.

$$H_i(x^*_{-i}, x_i) \leq H_i(x^*), \forall x_i, \quad P(x^*_{-i}, x_i) \leq P(x^*), \forall x_i$$

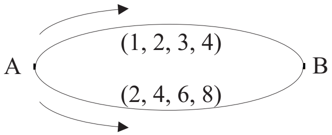
Potential games

Now, we argue that the game Γ' always has a pure strategy equilibrium. Let x^* be the pure strategy profile maximizing the potential $P(x)$ on the set $\prod_{i=1}^{n} X_i$. For any $x \in \prod_{i=1}^{n} X_i$, the inequality $P(x) \leq P(x^*)$ holds true at this point, particularly,

$$P(x^*_{-i}, x_i) \leq P(x^*), \forall x_i.$$

Therefore, x^* represents a Nash equilibrium in the game Γ' and, hence, in the game Γ.

Potential games



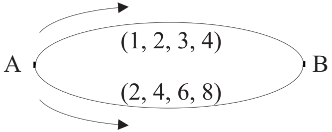
A game without potential. A game may have no potential, even if a pure strategy equilibrium does exist.

Suppose that the costs of players are defined by the maximal journey time of their packages on both links.

Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-4,-4) & (-3,-3) & (-2,-4) \\ (-3,-3) & (-4,-4) & (-6,-6) \\ (-4,-2) & (-6,-6) & (-8,-8) \end{pmatrix} \end{array}.$$

Potential games



The described game has no potential. We demonstrate this fact rigorously. Assume that a potential P exists; then:

$$P(1,1) - P(3,1) = H_1(1,1) - H_1(3,1) = -4 - (-4) = 0,$$

$$P(1,1) - P(1,2) = H_2(1,1) - H_2(1,2) = -4 - (-3) = -1.$$

And so,

$$P(3,1) - P(1,2) = -1.$$

On the other hand,

$$P(1,2) - P(3,2) = H_1(1,2) - H_1(3,2) = -3 - (-6) = 3,$$

$$P(3,1) - P(3,2) = H_2(3,1) - H_2(3,2) = -2 - (-6) = 4,$$

whence it follows that $P(3,1) - P(1,2) = 1$. This two facts contradicts, the game possesses no potential.

Definition. *A symmetrical congestion game is an n-player game* $\Gamma =< N, M, \{S_i\}_{i \in N}, \{c_i\}_{i \in N} >$, *where* $N = \{1, ..., n\}$ *stands for the set of players, and* $M = \{1, ..., m\}$ *means the set of some objects for strategy formation. A strategy of player* i *is the choice of a certain subset from* M. *The set of all feasible strategies makes the strategy set of player* i, *denoted by* S_i, $i = 1, ..., n$. *Each object* $j \in M$ *is associated with a function* $c_j(k), 1 \leq k \leq n$, *which represents the payoff (or costs) of each player from* k *players that have selected strategies containing* j. *This function depends only on the total number* k *of such players.*

Congestion games

Players have chosen strategies $s = (s_1, ..., s_n)$. The payoff function of player i is determined by the total payoff on each object:

$$H_i(s_1, ..., s_n) = \sum_{j \in s_i} c_j(k_j(s_1, ..., s_n)).$$

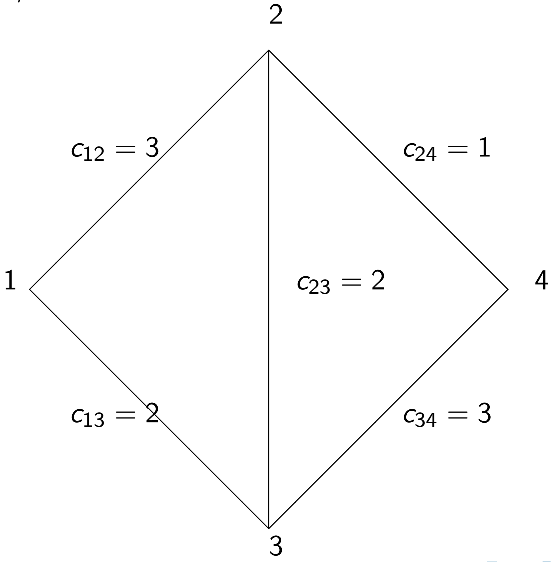
Here $k_j(s_1, ..., s_n)$ gives the number of players whose strategies incorporate object j, $i = 1, ..., n$.

Theorem. *A symmetrical congestion game is potential, ergo admits a pure strategy equilibrium.*

$$P(s_1, ..., s_n) = \sum_{j \in \cup_{i \in N} s_i} \left(\sum_{k=1}^{k_j(s_1, ..., s_n)} c_j(k) \right)$$

Gongestion game. Example

Players 1,2,3,4 send unit traffic via network from 1 to 4



Gongestion game. Example

Strategies: $s_1 = \{(1,2)(2,4)\}$, $s_2 = \{(1,3)(3,4)\}$,
$s_3 = \{(1,2)(2,3),(3,4)\}$, $s_4 = \{(1,3)(3,2),(2,4)\}$.
Calculate potential:

$$P(s_1,s_2,s_3,s_4) = \frac{1+2}{3} + \frac{1+2}{1} + \frac{1+2}{2} + \frac{1+2}{2} + \frac{1+2}{3} = 8$$
$$P(s_1,s_1,s_2,s_2) = \frac{1+2}{3} + \frac{1+2}{1} + \frac{1+2}{2} + \frac{1+2}{2} + \frac{1+2}{3} = 6.5$$
$$P(s_1,s_2,s_2,s_3) = \frac{1+2}{3} + \frac{1}{1} + \frac{1+2}{2} + \frac{1}{2} + \frac{1+2+3}{3} = 6.$$

Thus, profile (s_1,s_2,s_2,s_3) is Nash equilibrium.

Virtual operator market as a congestion game

There are m mobile network operators (MNO) $M = \{M_1, M_2, ..., M_m\}$. M_j is characterized by the parameters (p_j, m_j, r_j, c_j), $j = 1, ..., m$, where p_j - price for resource, m_j is number of consumers, r_j is the amount of a resource and c_j is a fee to join to this market.

There are n mobile network virtual operators (MNVO) $\{V_1, V_2, ..., V_n\}$ who compete in the market for the resources. We suppose that n is much larger than m. Each MNVO V_i has some private resource $v_i, i = 1, ..., n$. MNVO are the players in the game.

Our main achievements in this paper are:

1. Formulation of two-level competitive market for MNOs and MVNOs.
2. Finding market organization for identical players.
3. Analysing of pricing game in general
4. Finding equilibrium for allocation games in two-player case

Two-Stage Competition Model

Competition consists of two stages.
On the first stage the players (MNVO) choose some MNO to compete for the consumers. So, the strategy of player V_i is a subset $s_i \subset S_i$ from the feasible set $S_i \subseteq M$.

After the profile $s = (s_1, ..., s_n)$ is formed the players announce the prices for their service in each market j: $q_i^j, i = 1, ..., n; j \in s_i$.

Two-Stage Competition Model

The profile of the prices we denote $q = (q_1, ..., q_n)$. To avoid the monopoly we suppose that if the market M_j is occupied only one player V_i there is a restriction for the price $q_i \leq Q_j$. So, the payoff of the player V_i on the non-competitive market $M_j \in s_i$ is equal to

$$u_i^j(q) = (Q_j - p_j)m_j - c_j.$$

The payoff of player V_i on the competitive market $M_j \in s_i$ is equal to

$$u_i^j(q) = (q_i^j - p_j)m_j\gamma_i^j - c_j,$$

where γ_i^j is a proportion of consumers m_j who are interested in the service V_i.

Logistic law

Here we use the logistic function

$$\gamma_i^j = \frac{\exp\{-\alpha_{ij}q_i + \beta_{ij}v_i\}}{\sum\limits_{l:j\in s_l} \exp\{-\alpha_{lj}q_l + \beta_{lj}v_l\}\}} = \frac{k_{ij}\exp\{-\alpha_{ij}q_i\}}{\sum\limits_{l:j\in s_l} k_{lj}\exp\{-\alpha_{lj}q_l\}\}}, \quad j \in s_i,$$

where

$$k_{ij} = \exp\{\beta_{ij}v_i\}, \quad i = 1, ..., n, j \in s_i.$$

The general payoff of player V_i is the sum of payoffs in all used markets:

$$u_i(q) = \sum_{j\in s_i} u_i^j(q) = \sum_{j\in s_i'} \left((q_i^j - p_j)m_j\gamma_i^j - c_j\right) + \sum_{j\in s_i''} \left((Q_j - p_j)m_j - c_j\right),$$

$$i = 1, ..., n,$$

where s_i' and s_i'' are competitive and non-competitive markets, respectively.

The objective of the paper is to find equilibrium in the pricing model and then is to find the equilibrium in the allocation problem.

Model with Identical Players

Consider a case where all MNVO are identical, so all parameters $v_i, \alpha_{ij}, \beta_{ij}$ don't depend on the player i, and all markets are competitive. For simplicity assume that $k_{ij} = 1$. Consider the pricing game on the first market. Let $n_1 \geq 2$ players compete in this market. They announce the prices $q = (q_1, ..., q_{n_1})$. The payoff of player V_i is

$$u_i^1(q) = (q_i - p_1)m_1 \frac{\exp\{-\alpha_1 q_i\}}{\sum_{l=1}^{n_1} \exp\{-\alpha_1 q_l\}\}} - c_1, i = 1, ..., n_1.$$

Model with Identical Players

The first order condition for the equilibrium $\partial u_i(q)/\partial q_i = 0$ gives

$$\sum_{l=1}^{n_1} \exp\{-\alpha_1 q_l\} = (q_i - p_1) \sum_{l \neq i} \exp\{-\alpha_1 q_l\} \alpha_1.$$

By symmetry all prices in the equilibrium must be equal, for example q_1. It yields

$$q_1 = p_1 + \frac{1}{\alpha_1} \frac{n_1}{n_1 - 1}.$$

Model with Identical Players

Hence, the optimal payoff of player V_i on the first market is

$$u_i^1 = \frac{m_1}{\alpha_1} \frac{1}{n_1 - 1} - c_1, i = 1, ..., n_1. \tag{2}$$

We see that the payoff of any player is a decreasing function of number of players acted in this market. So, allocation game presented here is a congestion game [Rosental] which has a potential.

If the player V_i uses the allocation strategy s_i then his general payoff is

$$u_i = \sum_{j \in s_i} \left(\frac{m_j}{\alpha_j} \frac{1}{n_j(s) - 1} - c_j \right), i = 1, ..., n, \tag{3}$$

where $n_j(s)$ is the number of players chooses the market M_j (congestion vector).

Congestion game

To find equilibrium in the congestion game we can maximizing the potential function which has the following form

$$P(s_1, ..., s_n) = \sum_{j=1}^{m} \sum_{i=1}^{n_j(s)} \left(\frac{m_j/\alpha_j}{i-1} - c_j \right).$$

Consequently, the optimal allocation of the players among $\{M_1, ..., M_m\}$ can be found as a solution of the optimization problem

$$\sum_{j=1}^{m} \left(\frac{m_j}{\alpha_j} \sum_{i=1}^{n_j} \frac{1}{i-1} - c_j n_j \right) \to \max$$

in condition

$$\sum_{j=1}^{m} n_j = \sum_{i=1}^{n} |s_i|.$$

Congestion game

For example if the players can choose only one market then $\sum_{i=1}^{n} |s_i| = n$. For large n and small fee ($c_j \approx 0$) the optimization problem becomes

$$\sum_{j=1}^{m} \frac{m_j}{\alpha_j} \log n_j \to \max$$

the solution of which is

$$n_j \approx \frac{\frac{m_j}{\alpha_j}}{\sum_{l=1}^{m} \frac{m_l}{\alpha_l}} n, j = 1, ..., m,$$

so, for large n the players are allocated proportionally to the ratio of numbers of consumers and the weight of the player in the market. Notice that we don't know the precise location of the player but we know how many players will locate in each market.

Pricing Game

Consider the general case. Assume that the players are distributed among M_j in correspondence with the allocation rule $s = (s_1, ..., s_n)$. Consider a market M_j. In this market we have $n_j = n_j(s)$ players. For convenience let us re-enumerate players inside the market from 1 to n_j. Pricing game in the market M_j is a non-cooperative game on n_j players with payoff functions

$$u_i^j(q) = (q_i^j - p_j)m_j \frac{k_{ij} \exp\{-\alpha_{ij} q_i\}}{\sum\limits_{l:j\in s_l} k_{lj} \exp\{-\alpha_{lj} q_l\}\}} - c_j, i = 1, ..., n_j. \quad (4)$$

The game with these payoffs is a potential game and the NE $q^* = (q_1^*, ..., q_{n_j}^*)$ can be found as a maximum of potential

$$P_j(q) = \prod_{i=1}^{n_j} (q_i^j - p_j) \frac{\exp\{-\sum\limits_{l=1}^{n_j} \alpha_{lj} q_l\}}{\sum\limits_{l=1}^{n_j} k_{lj} \exp\{-\alpha_{lj} q_l\}}. \quad (5)$$

Pricing Game

The equilibrium q^* can be found from the first order condition $\partial P_j(q)/\partial q_i = 0, i = 1, ..., n_j$. It yields

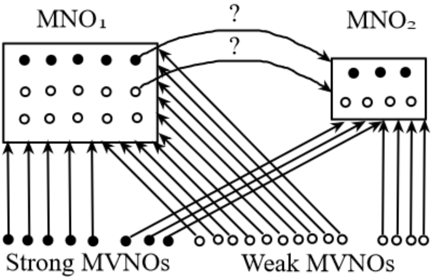
$$\sum_{l=1}^{n_j} k_{lj} \exp\{-\alpha_{lj} q_l^*\} = \alpha_{ij}(q_i^* - p_j)) \sum_{l=1 (l \neq i)}^{n_j} k_{lj} \exp\{-\alpha_{lj} q_l^*\}, i = 1, ..., n_j.$$

Suppose that a new player $n_j + 1$ join to the market M_j. New equilibrium prices $q' = (q'_1, ..., q'_{n_j}, q'_{n_j+1})$ satisfy the system of equations

$$\sum_{l=1}^{n_j+1} k_{lj} \exp\{-\alpha_{lj} q'_l\} = \alpha_{ij}(q'_i - p_j) \sum_{l=1 (l \neq i)}^{n_j+1} k_{lj} \exp\{-\alpha_{lj} q'_l\}, i = 1, ..., n_j+1.$$

Theorem. If a new player join to the market the payoffs of the players who compete in the market before it become less.

Allocation Game



.: Example of a game for 8 strong and 14 weak MVNOs

The optimal payoffs of the players in the equilibrium in market M_j depend not only on the number of players n_j, but also depend on the characteristic of the players (parameters α_{ij}, k_{ij}).

Stable allocation

Suppose that at the market there are two types of MVNOs. For example, there are "strong"and "weak"mobile virtual operators. Formally, it corresponds to the parameters $\alpha_{ij} = \alpha^1$, or $\alpha_{ij} = \alpha^2$, for all $j = 1, ..., m$, and $\alpha_1 \leq \alpha_2$.
Suppose that there are only two markets M_1 and M_2 and $n = n_1 + n_2$ mobile virtual operators where n_1, n_2 is the number of "strong"and "weak"MVNO. First of all, the players select a market. Then they play in "pricing game".

Stable allocation

Assume that the players are distributed in the following manner. On the market M_1 the distribution of "strong"and "weak"players is (k_1, k_2), and on the market M_2 the distribution is (l_1, l_2), $k_1 + l_1 = n_1$ and $k_2 + l_2 = n_2$. Let us find the equilibrium prices in each market. For simplicity assume that all parameters $k_{ij} = 1, \forall i, j$.

Stable allocation

Consider the market M_1. the price of MNO here is p_1. The profile of prices of MVNO is divided for two parts $q = (q_1^1, ..., q_{k_1}^1; q_1^2, ..., q_{k_2}^2)$, corresponding to "strong"and "weak players and the payoffs are

$$u_i^1(q) = (q_i^1 - p_1)m_1 \frac{\exp\{-\alpha^1 q_i^1\}}{\sum_{l=1}^{k_1} \exp\{-\alpha^1 q_l^1\} + \sum_{l=1}^{k_2} \exp\{-\alpha^2 q_l^2\}} - c_1, i = 1, ..., k_1,$$

for "strong"players and

$$u_i^2(q) = (q_i^2 - p_1)m_1 \frac{\exp\{-\alpha^2 q_i^2\}}{\sum_{l=1}^{k_1} \exp\{-\alpha^1 q_l^1\} + \sum_{l=1}^{k_2} \exp\{-\alpha^2 q_l^2\}} - c_2, i = 1, ..., k_2,$$

for "weak"players.

The first order condition for the equilibrium $\partial u_i^j(q)/\partial q_i = 0, \forall i, j = 1, 2$, and symmetry of players inside the groups yields that the equilibrium prices for "strong"and "weak players"are equal to q_1^*, q_2^*, respectively and satisfy the system of equation

$$(q_1 - p_1)\alpha^1 \left((k_1 - 1)\exp(-\alpha^1 q_1) + k_2 \exp(-\alpha_2 q_2)\right) =$$
$$k_1 \exp(-\alpha_1 q_1) + k_2 \exp(-\alpha_2 q_2) =$$
$$= (q_2 - p_1)\alpha^2 \left(k_1 \exp(-\alpha^1 q_1) + (k_2 - 1)\exp(-\alpha_2 q_2)\right) =$$

Hence, the optimal payoff of "strong"player on the first market is

$$u^1(k_1, k_2, m_1) = (q_1^* - p_1)\frac{m_1}{\alpha^1} \cdot \frac{\exp\{-\alpha^1 q_1^*\}}{(k_1 - 1)\exp\{-\alpha^1 q_1^*\} + k_2 \exp\{-\alpha^2 q_2^*\}} - c_1$$

and

$$u^2(k_1, k_2, m_1) = (q_2^* - p_1)\frac{m_1}{\alpha^2} \cdot \frac{\exp\{-\alpha^2 q_2^*\}}{k_1 \exp\{-\alpha^1 q_1^*\} + (k_2 - 1)\exp\{-\alpha^2 q_2^*\}} - c_2$$

for "weak"players.

Stable allocation

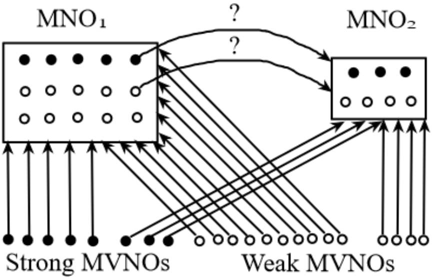
The same arguments are true for the equlibrium prices at the market M_2. Now we can determine when the allocation of n MVNOs among two markets M_1 and M_2 will be stable. Allocation $[(k_1, k_2); (l_1, l_2)]$ is **Nash-stable** if for each player it is not-profitable to deviate from the current market. Formally, it means that the following inequalities must be satisfied

$$u^1(k_1, k_2, m_1) \geq u^1(l_1{+}1, l_2, m_2), \quad u^2(k_1, k_2, m_1) \geq u^2(l_1, l_2{+}1, m_2),$$

$$u^1(k_1{+}1, k_2, m_1) \leq u^1(l_1, l_2, m_2), \quad u^2(k_1, k_2{+}1, m_1) \leq u^2(l_1, l_2, m_2).$$

Allocation Game

Consider the mobile network market with two MNOs, see Figure 2. The first market is large $m_1 = 1000$, the second is twice smaller $m_2 = 500$. There are twenty two MVNOs competing for the consumers at these markets. Suppose that among these MVNOs there are $n_1 = 8$ "strong"palyers and $n_2 = 14$ "weak"players, and $\alpha^1 = 1, \alpha^2 = 2$.



.: Example of a game for 8 strong and 14 weak MVNOs

Numerical example

Let the prices for the resource in both markets be equal $p_1 = p_2 = 1$, and the costs are $c_1 = 5, c_2 = 2$. Let us show that the allocation
$(k_1 = 5, k_2 = 10), (l_1 = 3, l_2 = 4)$ is Nash-stable.
We find the equilibrium prices in both markets. On the market M_1

$$q_1^* = 2.125, q_2^* = 1.523.$$

The payoffs of the both types players in the equlibrium on the market M_1 are

$$u^1(5, 10, 1000) = 120.299, u^2(5, 10, 1000) = 21.191.$$

On the market M_2 we find

$$q_1^* = 2.267, q_2^* = 1.550.$$

Numerical example

The payoffs of the both types players in the equlibrium on the market M_2 are

$$u^1(3, 4, 500) = 128.755, u^2(3, 4, 500) = 23.242.$$

We see that the market M_2 is more profitable for both types of players.

Prove the conditions for stability. Suppose, that a "strong"player from the market M_1 decides to move to the market M_2. We find that its payoff here is $u^1(4, 4, 500) = 102.598$. It is less than its payoff on the market M_1. So, it is not reasonable to move to another market. Now assume that the "weak player"moves from market M_1 to the market M_2. Its payoff here is $u^2(3, 5, 500) = 20.700$. It is less than on the market M_1. So, we see that the allocation $(k_1 = 5, k_2 = 10), (l_1 = 3, l_2 = 4)$ is Nash-stable.

Altman E., Shimkin N.: Individually optimal dynamic routing in a processor sharing system, Operation Research, pp. 776–784 (1998).

Aumann, R., Myerson, R.: Endogenous formation of links between players and coalitions: an application of the Shapley value. In: The Shapley value, Cambridge University Press, pp.175-191 (1988)

Avrachenkov, K., Litvak, N., Medyanikov, V., Sokol, M.: Alpha current flow betweenness centrality. In Proceedings of WAW 2013, LNCS v.8305, pp.106-117 (2013)

Avrachenkov, K., Litvak, N., Nemirovsky, D., Smirnova, E., Sokol, M.: Quick detection of top-k personalized pagerank lists. In Proceedings of WAW 2011, LNCS v.6732, pp.50-61 (2011)

Avrachenkov, K.E., Filar J.A., Howlett, P.G.: Analytic Perturbation Theory and its Applications, SIAM (2013).

Avrachenkov, K.E., Mazalov V.,V., Tsinguev B.T. : Beta Current Flow Centrality for Weighted Networks // In Proceedings of Computational Social Network, LNCS, v. 9197, pp. 216-227 (2015)

Awerbuch B., Azar Y., Epstein A.: The price of routing unsplittable flow, Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC 2005), pp. 331–-337 (2005).

Borgatti, S.P., Everett, M.G., Freeman, L.C.: Ucinet for Windows: Software for Social Network Analysis. Harvard (2002)

Borm, P., Owen, G., Tijs, S.: On the position value for communication situations. SIAM J. on Disc. Math., v.5(3), pp.305-320 (1992)

- Borm, P., van den Nouweland, A., Tijs, S.: Cooperation and communication restrictions: a survey. In: Imperfections and Behavior in Economic Organizations. Kluwer (1994)
- Braess D.: Uber ein Paradoxon der Verkehrsplanung, Unternehmensforschung **12**, pp. 258-–268 (1968).
- Brandes, U.: A faster algorithm for betweenness centrality. Journal of Mathematical Sociology, v.25, pp.163-177 (2001)
- Brandes, U., Fleischer, D.: Centrality measures based on current flow. In Proceedings of the 22nd annual conference on Theoretical Aspects of Computer Science, pp.533-544 (2005)
- Calvo, E., Lasaga, J., van den Nouweland, A.: Values of games with probabilistic graphs. Math. Social Sci., v.37, pp.79-95 (1999)
- Christodoulou G., Koutsoupias E. The price of anarchy of finite congestion games, Proc. of 37th annual ACM Symposium on Theory of Computing (STOC 2005), pp. 67–73 (2005)

Christodoulou G., Koutsoupias E. On the price of anarchy and stability of correlated equilibria of linear congestion games, LNCS, v. 3669, pp. 59–70 (2005)

Estrada E., Hatano N. Resistance Distance, Information Centrality, Node Vulnerability and Vibrations in Complex Networks, Network Science. Complexity in Nature and Technology. Springer. P. 13-29 (2010)

Freeman, L.C.: A set of measures of centrality based on betweenness. Sociometry, v.40, pp.35-41 (1977)

Freeman L.C., Borgatti S.P., White D.R. Centrality in valued graphs: A measure of betweenness based on network flow, Social Networks 13. P. 141-154 (1991). DOI:10.1016/0378-8733(91)90017-N

Gairing M., Monien B., Tiemann K. Routing (un-) splittable flow in games with player-specific linear latency functions.

Proc. of the 33rd International Colloquium on Automata Languages and Programming (ICALP 2006) , 501–512 (2006)

Hines P., Blumsack S. A centrality measure for electrical networks. In Proceedings of the 41st Hawaii International Conference on System Sciences. P. 185 (2008). DOI: 10.1109/HICSS.2008.5

Jackson, M.O.: Allocation rules for network games. Games and Econ. Behav., v.51(1), pp.128-154 (2005)

Jackson, M.O., Wolinsky, J.: A strategic model of social and economic networks. J. Econ. Theory, v.71(1), pp.44-74 (1996)

Jackson, M.O.: Social and economic networks. Princeton University Press (2008)

Jamison, R.E.: Alternating Whitney sums and matchings in trees. Part 1. Disc. Math., v.67, pp.177-189 (1987)

Korillis Y. A., Lazar A. A., Orda A. Avoiding the Braess's paradox for traffic networks, J. Appl. Probability **36**, pp. 211–222 (1999).

Lin H., Roughgarden T., Tardos E. On Braess's paradox, Proceedings of the 15th Annual ACM-SIAM Symp. on Discrete Algorithms (SODA04), pp. 333–-334 (2004).

Mavronicolas M., Spirakis P. The price of selfish routing, Proceedings of the 33th Annual ACM STOC (2001), pp. 510–-519 (2001)

Mazalov, V.: Mathematical Game Theory and Applications. Wiley (2014)

Mazalov, V.V., Trukhina, L.I.: Generating functions and the Myerson vector in communication networks. Disc. Math. and Appl. v.24(5), pp.295-303 (2014)

Milchtaich I. Congestion games with player-specific payoff functions, Games and Economic Behavior **13**, pp. 111–124 (1996)

Monderer D., Shapley L. Potential games, Games and Economic Behavior **14**, pp. 124–143 (1996)

Myerson, R.B.: Graphs and cooperation in games. Math. Oper. Res., v.2, pp.225-229 (1977)

Newman, M.E.J.: A measure of betweenness centrality based on random walks. Social networks, v.27, pp.39-54 (2005)

Opsahl, T., Agneessens, F., Skvoretz, J.: Node centrality in weighted networks: generalizing degree and shortest paths. Social Networks, v.32, pp.245-251 (2010)

Papadimitriou C.H., Koutsoupias E. Worst-Case Equilibria, LNSC, v. 1563, pp. 404–-413 (1999)

Papadimitriou C.H. Algorithms, games, and the Internet, Proceedings of the 33th Annual ACM STOC (2001), pp. 749–-753 (2001)

Roughgarden T., Tardos E. How Bad is Selfish Routing?, JACM, (2002)

Rosenthal R. W. A class of games possessing pure-strategy Nash equilibria, Int. Journal of Game Theory **2**, pp. 65–67 (1973)

Slikker, M.: Link monotonic allocation schemes. Int. Game Theory Review, vol. 7(4), pp.473-489 (2005)

Slikker, M., Gilles, R.P., Norde, H., Tijs, S.: Directed networks, allocation properties and hierarchy formation. Math. Social Sci., v.49(1), pp.55-80 (2005)

Talman, D., Yamamoto, Y.: Average tree solutions and subcore for acyclic graph games. J. Oper. Res. Soc. Japan, v.51(3), pp.187-201 (2008)

? Wang H., Hua H., Wang D. Cacti with minimum, second-minimum, and third-minimum Kirchhoff indices. Mathematical Communications. Vol. 15. P. 347-358 (2010)

? Wang Z., Scaglione A., Thomas R. Electrical centrality measures for electric power grid vulnerability analysis. Proceedings of the 49th IEEE Conference on Decision and Control (CDC). P. 5792 –5797 (2010). DOI: 10.1109/CDC.2010.5717964

Wardrope J.G. Some theoretical aspects of road traffic research, Proceedings of the Inst. Civil Engineers, pp. 325–-378 (1952)