On Game Theoretic Strategic Thinking

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Two-person zero-sum game in normal form

Definition 1

The system

$$\Gamma = (X, Y, K), \tag{1}$$

where X and Y are nonempty sets, and the function $K : X \times Y \to R^1$, is called a two-person zero-sum game in normal form.

The elements $x \in X$ and $y \in Y$ are called the *strategies* of players 1 and 2, respectively, in the game Γ, the elements of the Cartesian product $X \times Y$ (i.e. the pairs of strategies (x, y), where $x \in X$ and $y \in Y$) are called *situations*, and the function K is the payoff of Player 1. Player 2's payoff in situation (x, y) is equal to $[-K(x, y)]$; therefore the function K is also called the *payoff function* of the game Γ and the game Γ is called a *zero-sum game*. Thus, in order to specify the game Γ, it is necessary to define the sets of strategies X, Y for players 1 and 2, and the payoff function K given on the set of all situations $X \times Y$.

Two-person zero-sum game in normal form

The game Γ is interpreted as follows. Players simultaneously and independently choose strategies $x \in X, y \in Y$. Thereafter Player 1 receives the payoff equal to $K(x, y)$ and Player 2 receives the payoff equal to $(-K(x, y))$.

Definition 2

The game $\Gamma' = (X', Y', K')$ is called a subgame of the game $\Gamma = (X, Y, K)$ if $X' \subset X, Y' \subset Y$, and the function $K' : X' \times Y' \to R^1$ is a restriction of function K on $X' \times Y'$.

Definition 3

Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.

The Battle of the Bismarck See

Example 1. The conflict can be modeled as the following 2×2 matrix game

$$\begin{array}{cc} & \begin{array}{cc} N & S \end{array} \\ \begin{array}{c} N \\ S \end{array} & \left[\begin{array}{cc} 2 & 2 \\ 1 & 3 \end{array}\right]. \end{array}$$

The first player is US Admiral Kenney and the second Japanese Admiral Imamura. The conflict happens in the South Pacific in 1943. Imamura has to transport troops across the Bismarck See to New Guinea, and his opponent Kenney wants to bomb the transport. Imamura has two possible choices: a shorter Northern route (N, 2 days) or a longer Southern route (S, 3 days). Kenney must choose one of this routs (N or S) to send his planes to. If he chooses the wrong route he can call back the planes and send them to another route, but the number of bombing days is reduced by 1. We assume, that the number of bombing days represents the payoff to Kenney in a positive sense to Imamura in negative sense.

Maximin and minimax strategies

$$\underline{v} = \sup_{x \in X} \inf_{y \in Y} K(x, y), \tag{2}$$

is called the *lower value* of the game. The number

$$\overline{v} = \inf_{y \in Y} \sup_{x \in X} K(x, y) \tag{3}$$

is called the *upper value of the game* Γ.
Consider the $(m \times n)$ matrix game Γ_A. Then the extrema in (2) and (3) are reached and the lower and upper values of the game are, respectively equal to

$$\underline{v} = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij},$$

$$\overline{v} = \min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij}.$$

Maximin and minimax strategies

The minimax and maximin for the game Γ_A can be found by the following scheme

$$\left.\begin{matrix} \left[\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix}\right] & \begin{matrix} \min_j a_{1j} \\ \min_j a_{2j} \\ \dots \\ \min_j a_{mj} \end{matrix} \end{matrix}\right\} \max_i \min_j a_{ij}$$

$$\underbrace{\max_i a_{i1} \quad \max_i a_{i2} \quad \dots \quad \max_i a_{in}}_{\min_j \max_i a_{ij}}$$

Maximin and minimax strategies

Thus, in the game Γ_A with the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 5 & 3 & 8 \\ 6 & 0 & 1 \end{bmatrix}$$

the lower value (maximin) $\underline{v}$ and the maximin strategy i_0 of the first player are $\underline{v} = 3$, $i_0 = 2$, respectively, and the upper value (minimax) $\overline{v}$ and the minimax strategy j_0 of the second player are $\overline{v} = 3$, $j_0 = 2$, respectively.

Definition 4

In the two-person zero-sum game $\Gamma = (X, Y, K)$ the point (x^*, y^*) is called an equilibrium point, or a saddle point, if

$$K(x, y^*) \leq K(x^*, y^*), \tag{4}$$

$$K(x^*, y) \geq K(x^*, y^*) \tag{5}$$

for all $x \in X$ and $y \in Y$.

Mixed extension of a game

Consider the matrix game Γ_A. If the game has a saddle point, then the minimax is equal to the maximin; and each of the players can, by the definition of the saddle point, inform the opponent of his optimal (maximin, minimax) strategy and hence no player can receive extra benefits. Now assume that the game Γ_A has no saddle point. Then, we have

$$\min_j \max_i \alpha_{ij} - \max_i \min_j a_{ij} > 0. \tag{6}$$

In this case the maximin and minimax strategies are not optimal. Moreover, it is not advantageous for the players to play such strategies, as he can obtain a larger payoff. The information about a choice of a strategy supplied to the opponent, however, may cause greater losses than in the case of the maximin or minimax strategy.

Mixed extension of a game

Indeed, let the matrix A be of the form

$$A = \begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix}.$$

For this a matrix $\min_j \max_i \alpha_{ij} = 5$, $\max_i \min_j \alpha_{ij} = 3$, i.e. the saddle point does not exist. Denote by i^* the maximin strategy of Player 1 ($i^* = 1$), and by j^* the minimax strategy of Player 2 ($j^* = 2$). Suppose Player 2 adopts strategy $j^* = 2$ and Player 1 chooses strategy $i = 2$. Then the latter receives the payoff 5, i.e. 2 units greater than the maximin. If, however, Player 2 guesses the choice by Player 1, he alters his strategy to $j = 1$ and then Player 1 receives a payoff of 2 units only, i.e. 1 unit less than in the case of the maximin. Similar reasonings apply to the second player.
How to keep the information about the choice of the strategy in secret from the opponent? To answer this question, it may be wise to choose the strategy using some random device. In this case the opponent cannot learn the player's particular strategy in advance, since the player does not know it himself until strategy will actually be chosen at random.

Mixed extension of a game

Definition 5

The random variable whose values are strategies of a player is called a mixed strategy of the player.

Thus, for the matrix game Γ_A, a mixed strategy of Player 1 is a random variable whose values are the row numbers $i \in M$, $M = \{1, 2, \dots, m\}$. A similar definition applies to Player 2's mixed strategy whose values are the column numbers $j \in N$ of the matrix A.
Considering the above definition of *mixed strategies*, the former strategies will be referred to as *pure strategies*. Since the random variable is characterized by its distribution, the mixed strategy will be identified in what follows with the probability distribution over the set of pure strategies. Thus, Player 1's mixed strategy x in the game is the m-dimensional vector

$$x = (\xi_1, \dots, \xi_m), \ \sum_{i=1}^{m} \xi_i = 1, \xi_i \geq 0, i = 1, \dots, m. \tag{7}$$

Mixed extension of a game

Definition 6

The pair (x, y) of mixed strategies in the matrix game Γ_A is called the situation in mixed strategies.

We shall define the payoff of Player 1 at the point (x, y) in mixed strategies for the $(m \times n)$ matrix game Γ_A as the mathematical expectation of his payoff provided that the players use mixed strategies x and y, respectively. The players choose their strategies independently; therefore the mathematical expectation of payoff $K(x, y)$ in mixed strategies $x = (\xi_1, \dots, \xi_m)$, $y = (\eta_1, \dots, \eta_n)$ is equal to

$$K(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \xi_i \eta_j = (xA)y = x(Ay). \quad (8)$$

Mixed extension of a game

Definition 7

The point (x^*, y^*) in the game $\overline{\Gamma}_A$ forms a saddle point and the number $v = K(x^*, y^*)$ is the value of the game $\overline{\Gamma}_A$ if for all $x \in X$ and $y \in Y$

$$K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y). \tag{9}$$

Search game

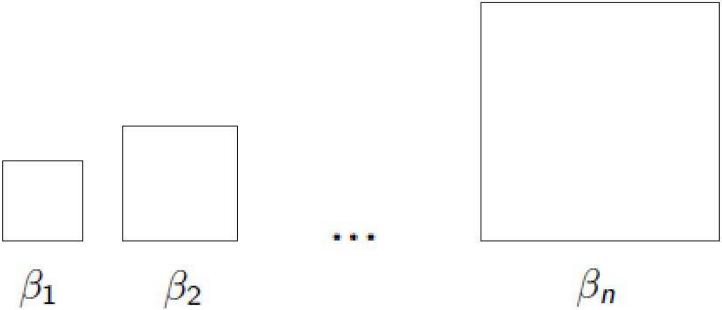
Example 2. This game is a special case of previous example, where matrix A has the form

$$A = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_n \end{bmatrix},$$

$$1 > \beta_1 > \beta_2 > ... > \beta_n > 0.$$

First player is called Searcher (S), second player Hider (H). The game proceeds as follows. Both players simultaneously choose one of boxes from the set $N = \{1, ..., n\}$. If H chooses box $i \in N$, he hides in the box i, if S chooses the box $j \in N$, he searches in the box j. If Searcher searches in the box $i \in N$, if Hider hides in $j \in N$ and $i \neq j$, the payoff of both players is equal to 0. If $i = j$, the Searcher wins β_i (the probability to find hider in the box under condition that Hider is there). In this game $\underline{v} = 0$, and $\overline{v} = \beta_n$ ($\underline{v} \neq \overline{v}$), and there is only mixed strategy saddle point. Denote optimal mixed strategies of players by $\overline{x} = (\overline{\xi_1}, ..., \overline{\xi_i}, ..., \overline{\xi_n})$ and $\overline{y} = (\overline{\eta_1}, ..., \overline{\eta_j}, ..., \overline{\eta_n})$.

Search game



Search game

Search game

Then if v is the value of the game we must have

$$\overline{\xi_i}\beta_i \geq v \geq \overline{\eta_i}\beta_i, \ i \in N.$$

Suppose that $\overline{\xi_i} > 0$, $\overline{\eta_i} > 0$, $i \in N$. Then we have

$$\overline{\xi_i}\beta_i = v = \overline{\eta_i}\beta_i,$$

$$\overline{\xi_i} = \frac{v}{\beta_i}, \ \overline{\eta_i} = \frac{v}{\beta_i},$$

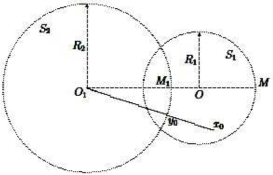
since $\sum_{i=1}^{n} \overline{\xi_i} = \sum_{i=1}^{n} \overline{\eta_i} = 1$, we get

$$\overline{\xi_i} = \frac{\frac{1}{\beta_i}}{\sum_{k=1}^{n} \frac{1}{\beta_k}} = \overline{\eta_i}, \ and \ v = \frac{1}{\sum_{k=1}^{n} \frac{1}{\beta_k}}.$$

ϵ-saddle points, ϵ-optimal strategies

Example 3. Suppose the sets S_1 and S_2 are the closed circles of radii R_1 and R_2 ($R_1 < R_2$), respectively. The payoff function $K(x, y) = \rho(x, y)$, where $\rho(x, y)$ is the Euclidean distance between x and y. Find the lower value of the game

$$\underline{v} = \max_{x \in S_1} \min_{y \in S_2} \rho(x, y).$$



Lower value of the game

Let $x_0 \in S_1$. Then $\min_y \rho(x_0, y)$ is achieved at the intersection point y_0 of the straight line, passing through the center O_1 of the circle S_2 and the point x_0, and the boundary of the circle S_2. Evidently, the quantity $\min_{y \in S} \rho(x_0, y)$ is a maximum at the point $M \in S_1$ where the lines of centers OO_1 intersect the boundary of the circle S_1 that is farthest from the point O_1. Thus, $\underline{v} = |O_1 M| - R_2$.

In order to compute the upper value of the game

$$\overline{v} = \min_{y \in S_2} \max_{x \in S_1} \rho(x, y)$$

we shall consider two cases.

Case 1. The center O of the circle S_1 belongs to the set S_2. For each $y_0 \in S_2$ the point x_0 providing $\max_{x \in S_1} \rho(x, y_0)$ is constructed as follows. Let x_0^1 and x_0^2 be the intersection points of the line $O_1 y_0$ and the boundary of the circle S_1, x_0^3 is the intersection point of the line Oy_0 with the boundary of the circle S_1, that is farthest from the point y_0. Then x_0 is determined from the condition

$$\rho(x_0, y_0) = \max_{i=1,2,3} \rho(x_0^i, y_0).$$

By construction, for all $y_0 \in S_2$

$$\max_{x \in S_1} \rho(x, y_0) = \rho(x_0, y_0) \geq R_1.$$

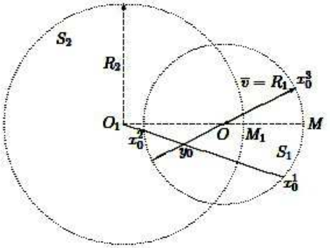
ϵ-saddle points, ϵ-optimal strategies

With $y_0 = O$, however, we get

$$\max_{x \in S_1} \rho(x, O) = R_1,$$

hence

$$\min_{y \in S_2} \max_{x \in S_1} \rho(x, y) = \overline{v} = R_1.$$



Case 1, when $O \in S_2$.

It can be readily seen that since $O \in S_2$, in Case 1 $\overline{v} = R_1 \geq |\overline{O_1 M}| - R_2 = \underline{v}$. Furthermore, we get the equality provided O belongs to the boundary of the set S_2.

ϵ-saddle points, ϵ-optimal strategies

Thus, if in Case 1 the point O does not belong to the boundary of the set S_2, then the game has no saddle point. If, however, the point O belongs to the boundary of the set S_2, then there exists a saddle point and an optimal strategy for Player 1 is to choose the point M lying at the intersection of the line of centers OO_1 with the boundary of the set S_1 that is farthest from the point O_1. An optimal strategy for Player 2 is to choose the point $y \in S_2$ coinciding with the center O of the circle S_1. In this case the value of the game is $v = \underline{v} = \overline{v} = R_1 + R_2 - R_2 = R_1$.

Case 2. The center of the circle $O \notin S_2$. This case is coinsidered in the same way as Case 1 when the center of the circle S_1 belongs to the boundary of the set S_2. Compute the quantity $\overline{v}$ (Fig. 2.4). Let $y_0 \in S_2$. Then the point x_0 providing $\max_{x \in S_1} \rho(x, y_0)$ coincides with the intersection point x_0 of the straight line, passing through y_0 and the center O of the circle S_1, and the boundary of the circle S_1 that is farthest from the point y_0. Indeed, the circle of radius $\overline{x_0 y_0}$ with its center at the point y_0 contains S_1 and its boundary is tangent to the boundary of the circle S_1 at the unique point x_0. Evidently, the quantity $\max_{x \in S_1} \rho(x, y) = \rho(x_0, y)$ takes its minimum value at the intersection point M_1 of the line segment OO_1 and the boundary of the circle S_2. Thus, in the case under study

$$\overline{v} = \min_{y \in S_2} \max_{x \in S_1} \rho(x, y) = |\overline{O_1 M}| - R_2 = \underline{v}.$$

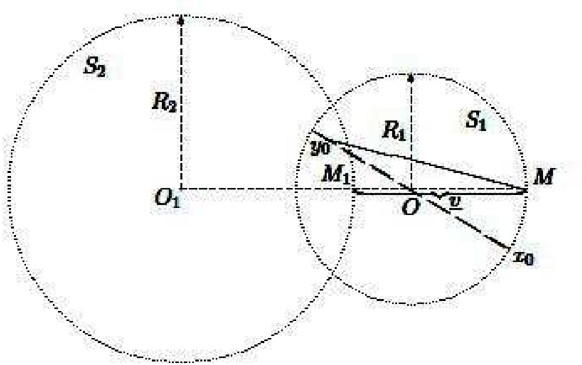
Optimal strategies for players 1 and 2 are to choose the points $M \in S_1$ and $M_1 \in S_2$, respectively.

ϵ-saddle points, ϵ-optimal strategies

If the open circles S_1 and S_2 are considered to be the strategy sets in Example 3, then in Case 2 the value of the game exists and is equal to

$$\underline{v} = \sup_{x \in S_1} \inf_{y \in S_2} \rho(x, y) = \inf_{y \in S_2} \sup_{x \in S_1} \rho(x, y) = \overline{v} = |\overline{O_1 M}| - R_2 = v.$$

Optimal strategies, however, do not exist, since $M \notin S_1$, $M_1 \notin S_2$. Nevertheless for any $\epsilon > 0$ there are ϵ-optimal strategies that are the points from the ϵ-neighborhood of the points M and M_1 belonging respectively to the sets S_1 and S_2.



Case 2, when $O \notin S_2$.

Definition of noncooperative game in normal form

Definition 8

The system

$$\Gamma = (N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N}),$$

where $N = \{1, 2, \dots, n\}$ is the set of players, X_i is the strategy set for player i, H_i is the payoff function for player i defined on Cartesian product of the players' strategy sets $X = \prod_{i=1}^{n} X_i$ (the set of situations in the game), is called a noncooperative game.

The noncooperative game Γ played by two players is called a *two-person game*.
The noncooperative two-person game Γ is then defined by the system $\Gamma = (X_1, X_2, H_1, H_2)$, where X_1 is the strategy set of one player, X_2 is the strategy set of the other player, $X_1 \times X_2$ is the set of situations, while $H_1 : X_1 \times X_2 \to R^1$, $H_2 : X_1 \times X_2 \to R^1$ are the payoff functions to the players 1 and 2, respectively. The finite noncooperative two-person game is called *bimatrix game*.

Definition of noncooperative game in normal form

This is due to the fact that, once the pure strategy sets of players have been designated by the numbers $1, 2, \dots, \overline{m}$ and $1, 2, \dots, \overline{n}$, the payoff functions can be written in the form of two matrices

$$H_1 = A = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1\bar{n}} \\ \dots & \dots & \dots \\ \alpha_{\bar{m}1} & \dots & \alpha_{\bar{m}\bar{n}} \end{bmatrix} \text{ and } H_2 = B = \begin{bmatrix} \beta_{11} & \dots & \beta_{1\bar{n}} \\ \dots & \dots & \dots \\ \beta_{\bar{m}1} & \dots & \beta_{\bar{m}\bar{n}} \end{bmatrix}.$$

Here the elements α_{ij} and β_{ij} of the matrices A, B are respectively the payoffs to players 1 and 2 in the situation (i, j), $i \in \overline{M}$, $j \in \overline{N}$, $\overline{M} = \{1, \dots, \overline{m}\}$, $\overline{N} = \{1, \dots, \overline{n}\}$.

Definition of noncooperative game in normal form

In line with the foregoing, the bimatrix game is played as follows. Player 1 chooses number i (the row) and Player 2 (simultaneously and independently) chooses number j (the column). Then Player 1 receives the amount $\alpha_{ij} = H_1(x_i, y_j)$ and Player 2 receives the amount $\beta_{ij} = H_2(x_i, y_j)$.

Note that the bimatrix game with matrices A and B can also be described by the $(\overline{m} \times \overline{n})$ matrix (A, B), where each element is a pair $(\alpha_{ij}, \beta_{ij})$, $i = 1, 2, \dots, \overline{m}$, $j = 1, 2, \dots, \overline{n}$. The game determined by the matrix A and B will be denoted as $\Gamma(A, B)$.

If the noncooperative two-person game Γ is such that $H_1(x, y) = -H_2(x, y)$ for all $x \in X_1$, $y \in X_2$, then Γ appears to be a zero-sum two-person game discussed in the preceding chapters. In the special bimatrix game, where there is $\alpha_{ij} = -\beta_{ij}$, we have a matrix game.

Battle of the sexes.

Example 4. Consider the bimatrix game determined by

$$(A, B) = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \left[\begin{array}{cc} (4,1) & (0,0) \\ (0,0) & (1,4) \end{array} \right] \end{array}.$$

Although this game has a variety of interpretations, the best known seems to be the following. Husband (Player 1) and his wife (Player 2) may choose one of two evening entertainments: football match (α_1, β_1) or theater (α_2, β_2). If they have different desires, (α_1, β_2) or (α_2, β_1), they stay at home. The husband shows preference to the football match, while his wife prefers to go to the theater. However, it is more important for them to spend the evening together than to be alone at the entertainment (though preferable).

"Crossroads" game.

Example 5. Two drivers move along two mutually perpendicular routes and simultaneously meet each other at a crossroad. Each driver may make a stop (1st strategy, α_1 or β_1) or continue on his way (2nd strategy, α_2 or β_2).
It is assumed that each player prefers to make a stop in order to avoid an accident, or to continue on his way if the other player has made a stop.
This conflict can be formalized by the bimatrix game with the matrix

$$(A, B) = \begin{matrix} \\ \alpha_1 \\ \alpha_2 \end{matrix} \begin{matrix} \beta_1 & \beta_2 \\ \left[\begin{matrix} (1,1) & (1-\epsilon, 2) \\ (2, 1-\epsilon) & (0,0) \end{matrix} \right. & \left. \right] \end{matrix}$$

(the non-negative number ϵ corresponds to the feeling of dissatisfaction that one player has to make a stop and let the other go).

Optimality principles in noncooperative games

Definition 9

The situation $x^* = (x_1^*, \dots, x_i^*, \dots, x_n^*)$ is called the Nash equilibrium if for all $x_i \in X_i$ and $i = 1, \dots, n$ there is

$$H_i(x^*) \geq H_i(x^* \| x_i). \tag{10}$$

Example 6. Prisoners' dilemma. Consider the bimatrix game with payoffs

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (5,5) & (0,10) \\ (10,0) & (1,1) \end{bmatrix} \end{array}.$$

Here we have one equilibrium situation (α_2, β_2) (though not strong equilibrium), which yields the payoff vector $(1, 1)$. However, if both players play (α_1, β_1), they obtain the payoffs $(5, 5)$, which is better to both of them. Zero-sum games have no such paradoxes. As for this particular case, the result is due to the fact that a simultaneous deviation from the equilibrium strategies may further increase the payoff to each player.

Optimality principles in noncooperative games

Example 7. Braess's paradox. The model was proposed by D. Braess (1968).

Suppose that 60 cars (players) move from point A to point B. The time for passing from C to B and from A to D equals 60 min. (and does not depend on number of cars on each of arcs AD and CB. On arcs AC and DB the passing time is equal to the number of cars using this arcs. Each player (car) from the set of 60 players (cars) has to go from A to B, and has the possibility to choose one of two roads (strategies) ACB or ADB. It is clear that

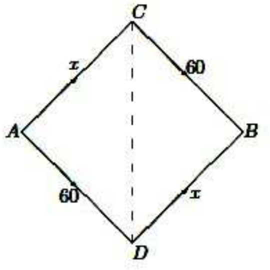


Figure: Braess's paradox

Nash equilibrium ujhikjh is such allocation of cars in which the time of passing along ACB is equal to the time passing along ADB.

Optimality principles in noncooperative games

If x is the number of cars using ACB and y is the number of cars using ADB we must have in Nash equilibrium

$$60 + x = 60 + y, \quad x + y = 60,$$

which gives us $x = y = 30$ (the proof is clear, since if one car changes his mind and switches, for instance, from road ACB to ADB he will need passing time $60 + 30 + 1 = 91$, but in Nash equilibrium his time is $60 + 30 = 90$).

Suppose now that we connect points C and D by speed way, which each car can pass in time 0. Then we see, that any car, which chooses ACB or ADB will benefit by moving along ACDB spending 60 instead of 90 along ADB or ACB. This means that allocation of cars along ACB and ADB will not be Nash equilibrium after opening a new road CD. It is easily seen, that the new Nash equilibrium in this case will be all cars use ACDB with passing time 120, since if one car deviates she will get the same amount $60 + 60 = 120$.

We observe a paradoxical case – the time passing from A to B increases from 90 to 120 after a new road construction.

Definition 10

The situation $\overline{x}$ in the noncooperative game Γ is called pareto-optimal if there is no situation $x \in X$ for which the following inequalities hold:

$$H_i(x) \geq H_i(\overline{x}) \text{ for all } i \in N \text{ and}$$

$$H_{i_0}(x) > H_{i_0}(\overline{x}) \text{ for at least one } i_0 \in N.$$

The set of all pareto-optimal situations will be denoted by X^P.

Games in characteristic function form

Let $N = \{1, \dots, n\}$ be a set of all players. Any nonempty subset $S \subset N$ is called a *coalition.*

Definition 11

The real-valued function v defined on coalitions $S \subset N$ is called a characteristic function of the n-person game. Here the inequality

$$v(T) + v(S) \leq v(T \cup S), \quad v(\oslash) = 0 \tag{11}$$

holds for any nonintersecting coalitions T, S ($T \subset N$, $S \subset N$).

Property (11) is called a *superadditivity property.* This property is necessary for the number $v(T)$ to be conceptually interpreted as a guaranteed payoff to a coalition T when this coalition is acting independently of other players. This interpretation of inequality (11) implies that the coalition $S \cup T$ has no less opportunities then the two nonintersecting coalitions S and T when they act independently.

From the superadditivity of v it follows that for any system of nonintersecting coalitions $S_1, \dots, S_k$ there is $\sum_{i=1}^{n} v(S_i) \leq v(N)$. This, in particular, implies that there is no decomposition of the set N into coalitions such that the guaranteed total payoff to these coalitions exceeds the maximum payoff to all players $v(N)$.

Investment fund

Three investment fund managers consider investment possibilities for a year. Fund manager 1 has \$3000000 to invest, manager 2 has \$1000000 and manager 3 has \$2000000. There is an investment scheme

	Deposit	Interest rate
1	less than \$2000000	8%
2	\$2000000 up to \$5000000	9%
3	\$5000000 and more	10%

This situation can be readily translated into a characteristic function v for a three-player game, $N = \{1, 2, 3\}$ (in \$10000 units):

$$v(N) = 60, \quad v(\{1,2\}) = 36, \quad v(\{1,3\}) = 50, \quad v(\{2,3\}) = 27,$$

$$v(\{1\}) = 27, \quad v(\{2\}) = 8, \quad v(\{3\}) = 18.$$

It's simply to test that characteristic function $v(S)$, $S \subset N$ has superadditivity property.

"Jazz band" game

Manager of a club promises singer S, pianist P, and drummer D to pay \$100 for a joint performance. He values a singer-pianist duet at \$80, a drummer-pianist duet at \$65 and a pianist at \$30.

A singer-drummer duet may earn \$50 and a singer, on the average, \$20 for doing an evening performance. A drummer may not earn anything by playing alone.

Designating players S, P, and D by numbers 1,2,3, respectively, we are facing a cooperative game (N, v), where $N = \{1, 2, 3\}$, $v(1, 2, 3) = 100$, $v(1, 3) = 50$, $v(1) = 20$, $v(1, 2) = 80$, $v(2, 3) = 65$, $v(2) = 30$, $v(3) = 0$.

Games in characteristic function form

In what follows, by the cooperative game is meant a pair (N, v), where v is the characteristic function satisfying inequality (11). The conceptual interpretation of the characteristic function justifying property (11) is not essential for what follows.

The main problem in the cooperative theory of n-person games is to construct realizable principles for optimal distribution of a maximum total payoff $v(N)$ among players.

Let α_i be an amount the player i receives by distribution of maximum total payoff $v(N)$, $N = \{1, 2, \dots, n\}$.

Definition 12

The vector $\alpha = (\alpha_1, \dots, \alpha_n)$, which satisfies the conditions

$$\alpha_i \geq v(\{i\}), \ i \in N, \tag{12}$$

$$\sum_{i=1}^{n} \alpha_i = v(N), \tag{13}$$

where $v(\{i\})$ is the value of characteristic function for a single-element coalition $S = \{i\}$ is called an imputation.

Condition (12) is called an individual rationality condition and implies that every member of coalition received at least the same amount he could ensure by acting alone, without any support of other players. Furthermore, condition (13) must be satisfied, since in the case $\sum_{i \in N} \alpha_i < v(N)$ there is a distribution α', on which every player $i \in N$ receives more than his share α_i. However, if $\sum_{i \in N} \alpha_i > v(N)$, then players from N distribute among themselves an unrealized payoff. For this reason, the vector α can be taken to be admissible only if condition (13) is satisfied. This condition is called a collective (or group) rationality condition.

By (12), (13), for the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ to be an imputation in the cooperative game (N, v), it is necessary and sufficient that it could be represented as

$$\alpha_i = v(\{i\}) + \gamma_i, \quad i \in N,$$

and

$$\gamma_i \geq 0, \; i \in N, \quad \sum_{i \in N} \gamma_i = v(N) - \sum_{i \in N} v(\{i\}).$$

Definition 13

The game (N, v) is called essential if

$$\sum_{i \in N} v(\{i\}) < v(N),$$

otherwise it is called nonessential.

For any imputation α, we denote the quantity $\sum_{i \in S} \alpha_i$ by $\alpha(S)$ and the set of all imputations by D. The nonessential game has a unique imputation $\alpha = (v(\{1\}), v(\{2\}), \ldots, v(\{n\}))$.
In any essential game with more than one player, the imputation set is infinite. We shall examine such games by using a dominance relation.

Definition 14

Imputation α dominates imputation β in coalition S (denoted as $\alpha \overset{S}{\succ} \beta$) if

$$\alpha_i > \beta_i,\ i \in S, \quad \alpha(S) \le v(S).$$

The first condition in (37) implies that imputation α is more advantageous to all members of coalition S than imputation β, while the second condition accounts for the fact that imputation α can be realized by coalition S (that is, coalition S can actually offer an amount α_i to every player $i \in S$).

Definition 15

Imputation α is said to dominanate imputation β if there is a coalition S for which $\alpha \overset{S}{\succ} \beta$. Dominance of imputation β by imputation α is denoted as $\alpha \succ \beta$.

Dominance is not possible in a single element coalition and in the set of all players N. Indeed, $\alpha \overset{i}{\succ} \beta$ had to imply $\beta_i < \alpha_i \leq v(\{i\})$ which contradicts condition (13).

The core and *NM*-solution

The following approach is possible. Suppose the players in the cooperative game (N, v) have come to an agreement on distribution of a payoff to the whole coalition N (imputation α^*), under which none of the imputations dominates α^*. Then such a distribution is stable in that it is disadvantageous for any coalition S to separate from other players and distribute a payoff $v(S)$ among its members. This suggests that it may be wise to examine the set of nondominant imputations.

Definition 16

The set of nondominant imputations in the cooperative game (N, v) is called core.

Then we have the theorem which characterizes core.

Theorem.

For the imputation α to belong to core, it is necessary and sufficient that

$$v(S) \leq \alpha(S) = \sum_{i \in S} \alpha_i$$

hold for all $S \subset N$.

The core and *NM*-solution in "jazz band" game

The total receipts of three musicians is maximum ($100) when they do performance jointly. If the singer does performance separately from the pianist and drummer, they receive $65+$20 all together. If the pianist does performance separately from the singer and drummer, they receive $30+$50 all together. Finally, if the pianist and singer do performance without the drummer, their total receipts amount to $80. What is the distribution of the maximum total receipts to be considered rational in terms of the above-mentioned partial cooperation and individual behavior?
The vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in the "jazz band" game belongs to the core if and only if

$$\begin{cases} \alpha_1 \geq 20, \ \ \alpha_2 \geq 30, \ \ \alpha_3 \geq 0, \\ \alpha_1 + \alpha_2 + \alpha_3 = 100, \\ \alpha_1 + \alpha_2 \geq 80, \ \ \alpha_2 + \alpha_3 \geq 65, \ \ \alpha_1 + \alpha_3 \geq 50. \end{cases}$$

This set is a convex hull of the following three imputations: (35,45,20), (35,50,15), (30,50,20). Thus, the payoffs of each player in different imputations differs on the amount not more than 5 rubles. The typical representative of the core is the arithmetical mean of extreme points of core, namely $\alpha^* = (33.3, 48.3, 18.3)$. The characteristic feature of the imputation a^* is that all two-component coalitions have the same additional receipts: $\alpha_i + \alpha_j - v(\{i,j\}) = 1.6$. The imputation α^* is a fair compromise from the interior of the core.

The Shapley value

The multiplicity of the previously discussed optimality principles (core and NM-solution) in cooperative games and the rigid conditions on the existence of these principles force us to a search for the principles of optimality, existence and uniqueness of which may be ensured in every cooperative game. Among such optimality principles is Shapley value. The Shapley value is defined axiomatically

Shapley axioms.

1. If S is any carrier of the game (N, v), then $\sum_{i \in S} \varphi_i[v] = v(S)$.
2. For any substitution of π and $i \in N$ $\varphi_{\pi(i)}[\pi v] = \varphi_i[v]$.
3. If (N, u) and (N, v) are any cooperative games, then

$$\varphi_i[u + v] = \varphi_i[u] + \varphi_i[v].$$

$$\varphi_i[v] = \sum_{T | i \in T \subset N} \frac{(t-1)!(n-t)!}{n!}[v(T) - v(T \setminus \{i\})].$$

Multistage games with perfect information

We shall now define the *multistage game with perfect information on a finite tree graph.*

Let $G = (X, F)$ be a tree graph. Consider the partition of the node set X into $n+1$ sets $X_1, \dots, X_n, X_{n+1}$, $\cup_{i=1}^{n+1} X_i = X$, $X_k \cap X_l = \oslash$, $k \neq l$, where $F_x = \oslash$ for $x \in X_{n+1}$. The set X_i, $i = 1, \dots, n$ is called the *set of personal positions* of the i-th player, while the set X_{n+1} is called the *set of final* or *terminal positions*. The real-valued functions $H_1(x), \dots, H_n(x)$, $x \in X_{n+1}$ are defined on the set of final positions X_{n+1}. The function $H_i(x)$, $i = 1, \dots, n$ is called a *payoff* to the i-th player.

The game proceeds as follows. Let there be given the set N of players designated by natural numbers $1, \dots, i, \dots, n$ (hereafter denoted as $N = \{1, 2, \dots, n\}$). Let $x_0 \in X_{i_1}$, then in the node (position) x_0 player i_1 "makes a move" and chooses the next node (position) $x_1 \in F_{x_0}$. If $x_1 \in X_{i_2}$, then in the node x_1 Player i_2 "makes a move" and chooses the next node (position) $x_2 \in F_{x_1}$ and so on. Thus, if the node (position) $x_{k-1} \in X_{i_k}$ is realized at the k-th step, then in this node Player i_k "makes a move" and selects the next node (position) from the set $F_{x_{k-1}}$. The game terminates as soon as the terminal node (position) $x_l \in X_{n+1}$, (i.e. the node for which $F_{x_l} = \oslash$) is reached.
Such a step-by-step selection implies a unique realization of some sequence $x_0, \dots, x_k, \dots, x_l$ determining the path in the tree graph G which emanates from the initial position and reaches one of the final positions of the game. In what follows, such a path is called a *play* or *path of the game.*

Multistage games with perfect information

Because of the tree-like structure of the graph G, each play uniquely determines the final position x_l to be reached and, conversely, the final position x_l uniquely determines the play. In the position x_l each of the players i, $i = 1, \dots, n$, receives a payoff $H_i(x_l)$.
We assume that Player i making his choice in position $x \in X_i$ knows this position and hence, because of the tree-like structure of the graph G, can restore all previous positions. In this case, the players are said to have perfect information. Chess and draughts provide a good example of the game with perfect information, because players record their moves, and hence they know the past course of the game when making each move in turn.

Multistage games with perfect information

Definition 17

The single-valued map u_i which sets up a correspondence between each node (position) $x \in X_i$ and some unique node (position) $y \in F_x$ is called a strategy for player i.

The set of all possible strategies for player i is denoted by U_i. Now the strategy of i-th player prescribes him, in any position x from his personal positions X_i, a unique choice of the next position.
The ordered set $u = (u_1, \dots, u_i, \dots, u_n)$, where $u_i \in U_i$, is called a *situation in the game*,
while the Cartesian product $U = \prod_{i=1}^{n} U_i$ is called the *set of situations*.
Each situation $u = (u_1, \dots, u_i, \dots, u_n)$ uniquely determines a play in the game, and hence payoffs of the players. Indeed, let $x_0 \in X_{i_1}$. In the situation $u = (u_1, \dots, u_i, \dots, u_n)$ the next position x_1 is then uniquely determined by the rule $u_{i_1}(x_0) = x_1$. Now let $x_1 \in X_{i_2}$. Then x_2 is uniquely determined by the rule $u_{i_2}(x_1) = x_2$. If the position $x_{k-1} \in X_{i_k}$ is realized at the k-th step, then x_k is uniquely determined by the rule $x_k = u_{i_k}(x_{k-1})$ and so on.

Suppose that the situation $u = (u_1, \ldots, u_i, \ldots, u_n)$ in the above sense determines a play $x_0, x_1, \ldots, x_l$. Then we may introduce the notion of the *payoff function* K_i of player i by equating its value in each situation u to the value of the payoff H_i in the final position of the play $x_0, \ldots, x_l$ corresponding to the situation $u = (u_1, \ldots, u_n)$, that is

$$K_i(u) = K_i(u_1, \ldots, u_i, \ldots, u_n) = H_i(x_l), \quad i = 1, \ldots, n.$$

Functions K_i, $i = 1, \ldots, n$, are defined on the set of situations $U = \prod_{i=1}^{n} U_i$. Thus, constructing the players' strategy sets U_i and defining the payoff functions K_i, $i = 1, \ldots, n$, on the Cartesian product of strategy sets of players we obtain a game in normal form

$$\Gamma = (N, \{U_i\}_{i \in N}, \{K_i\}_{i \in N}),$$

where $N = \{1, \ldots, i, \ldots, n\}$ is the set of players, U_i is the strategy set for player i, and K_i is the payoff function for player i, $i = 1, \ldots, n$.

Multistage games with perfect information

For the purposes of further investigation of the game Γ we need to introduce the notion of a *subgame*, i.e. the game on a subgraph of the graph G in the main game.

Let $z \in X$. Consider a subgraph $G_z = (X_z, F)$ which is associated with the subgame Γ_z as follows. The players personal positions in the subgame Γ_z are determined by the rule $Y_i^z = X_i \cap X_z$, $i = 1, \dots, n$, the set of final positions $Y_{n+1}^z = X_{n+1} \cap X_z$, player i's payoff $H_i^z(x)$ in the subgame is taken to be

$$H_i^z(x) = H_i(x), \; x \in Y_{n+1}^z, \; i = 1, \dots, n.$$

Accordingly, player i's strategy u_i^z in the subgame Γ_z is defined to be the truncation of player i's strategy u_i in the game Γ to the set Y_i^z, i.e.

$$u_i^z(x) = u_i(x), \; x \in Y_i^z = X_i \cap X_z, \; i = 1, \dots, n.$$

The set of all strategies for player i in the subgame is denoted by U_i^z. Then each subgraph G_z is associated with the subgame in normal form

$$\Gamma_z = (N, \{U_i^z\}, \{K_i^z\}),$$

where the payoff function K_i^z, $i = 1, \dots, n$ are defined on the Cartesian product $U^z = \prod_{i=1}^n U_i^z$.

Absolute equilibrium (subgame-perfect)

We introduced the notion of a Nash equilibrium for the n-person game in normal form. It turns out that for multistage games it is possible to strengthen the notion of equilibrium by introducing the notion of an absolute equilibrium.

Definition 18

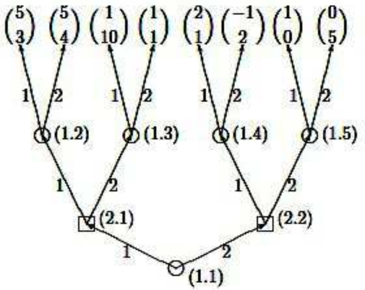
The Nash equilibrium $u^* = (u_1^*, \dots, u_n^*)$ is called an absolute Nash equilibrium in the game Γ if for any $z \in X$ the situation $(u^*)^z = ((u_1^*)^z, \dots, (u_n^*)^z)$, where $(u_i^*)^z$ is the truncation of strategy u_i^* to the subgame Γ_z, is Nash equilibrium in this subgame.

Then the following fundamental theorem is valid.

Theorem.

In any multistage game with perfect information on a finite tree graph there exists an absolute Nash equilibrium.

Example 8. We have seen in the previous example that "favorableness" of the players give them higher payoffs in the corresponding Nash equilibria, than the "unfavorable" behavior. But it is not always the case. Sometimes the "unfavorable" Nash equilibrium gives higher payoffs to all the players than "favorable" one. We shall illustrate this rather nontrivial fact on example. Consider the two-person game on the Figure.

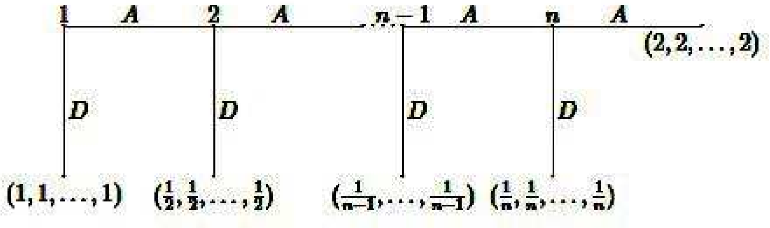


Absolute equilibrium (subgame-perfect)

The nodes from the personal positions X_1 are represented by circles and those from X_2 by blocks, with players payoffs written in final position. On the figure positions from the sets X_i $(i = 1, 2)$ are numbered by double indexes (i, j) where i is the index of the player and j the index of the node x in the set X_i. One can easily see that the "favorable" equilibrium has the form $((2, 2, 1, 1, 1), (2, 1))$ with payoffs $(2, 1)$. The "unfavorable" equilibrium has the form $((1, 1, 2, 1, 1), (1, 1))$ with payoffs $(5, 3)$.

Absolute equilibrium (subgame-perfect)

Example 9. Consider the n-person game with perfect information, where each player $i \leq n$ can either end the game by playing D or play A and give the move to player $i+1$.



Absolute equilibrium (subgame-perfect)

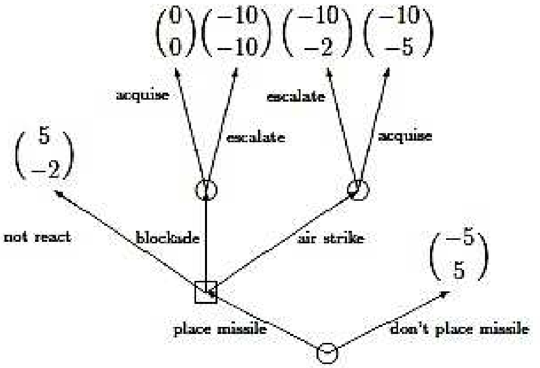
If player i selects D, each player gets $1/i$, if all players select A each gets 2. The backward induction algorithm for computing the subgame perfect (absolute) equilibria predicts that all players should play A. Thus the situation $(A, A, \ldots, A)$ is a subgame perfect Nash equilibrium. (Note that in the game under consideration each player is moving only once and has two alternatives, wich are also his strategies.) But there are also other equilibria. One class of Nash equilibria has the form $(D, A, A, D, \ldots)$, where the first player selects D and at least one of the others selects D. The payoffs in the first case are $(2, 2, \ldots, 2)$ and in the second $(1, 1, \ldots, 1)$. On the basis of robustness argument it seems that the equilibrium $(A, A, \ldots, A)$ is inefficient if n is very large. The equilibrium $(D, A, A, D, \ldots)$ is such because the player 4 uses the punishment strategy to enforce player 1 to play D. This equilibrium is not subgame perfect, because it is not an equilibrium in any subgame starting from the positions $2, 3$.

Cuban missile crises

Example 10. Consider now the example which can in a very simplified version be modelled by a game on the tree with perfect information. Namely the Cuban missile crises between the United States under John Kennedy and Soviet Union under Nikita Khrushchev in 1963. Khrushchev got information from his secret service that USA is planning a nuclear air over USSR planning to destroy 60 main USSR cities. To prevent this attack he starts the game by deciding whether or not to place intermediate range ballistic missiles in Cuba. If he places the missiles, his opponent player, Kennedy, will have three options: not react, blockade Cuba or eliminate the missiles by special airstrike. If Kennedy chooses the aggressive action of a blockade or an airstrike, Khrushchev may acquiesce or he may go by way of escalation with possible nuclear war at the end.

Cuban missile crises

Consider the game tree on figure.



In this game Khrushchev – the first player – moves in circled vertexes and Kennedy – the second player – in squared vertexes. Payoffs are written in the final vertexes and on the first place is the payoff of Khrushchev.

Interpret the payoffs. If Player 1 (Khrushchev) decides not to place missiles his payoff is (-5), since there will be the probability of nuclear strike against USSR, in this case Player 2 (Kennedy) will get (5) since at that time USSR did not have the opportunity to strike back with nuclear bombs on long distances. If Player 1 chooses on first stage place missiles, then the second player after some time will be informed and will have the possibility of taking one of following alternatives: not react (n), blockade Cuba (b), air strike to eliminate the missiles (a). If Player 2 chooses (n) then the game is over and the payoff of the Player 1 will be (5) and payoff of the Player 2 will be (-2). Since in this case the nuclear war will have a very small probability but strategic position of USSR would be much better than of USA. If Player 2 decides to blockade, then Player 1 has two alternatives: to acquire (a) or to escalate (e).

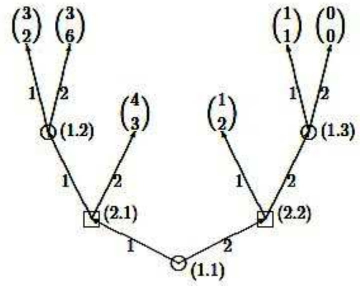
Cuban missile crises

In the case Player 1 chooses (a) the game is draw with payoffs $(0, 0)$, which means that USSR will not keep missiles in Cuba and USA will forget the idea of nuclear air strike against USSR after Cuba lesson. If Player 1 decides to escalate the nuclear war is possible and the payoffs will be $(-10, -10)$ the same for both, since compared with the case after 1 player alternative do not place missiles the nuclear war will make symmetric damage on both countries. Suppose now that Player 2 chooses air strike (alternative a), then if Player 1 chooses (a) the payoffs will be $(-10, -2)$ since in this case Player 2 will have incentive to start nuclear war (as he wanted before the crises) but maybe some missiles in Cuba will remain untouched by air strike and few USA cities may be destroyed by nuclear attack (-2). In the case if Player 1 choused (e), the payoff will be $(-10, -5)$ with less damage for USA since air strike can decrease the power of USSR missiles in Cuba (but not too much). Nash equilibrium can be found by backward induction and has payoff $(0, 0)$, which really happens.

Absolute equilibrium (subgame-perfect)

Indifferent equilibrium. As we have seen from Example (8) a subgame perfect equilibrium may appear non-unique in an extensive form game. This happens when the payoffs of some players coincides in terminal positions. Then the behavior of the player depends on his attitude to his opponents and the behavior of the player type naturally arises. In two person cases one way destinguish with between two types of players "favorable" and "unfavorable". The resulting two different subgame perfect Nash equilibrium where demonstrated in Example (8).

Absolute equilibrium (subgame-perfect)



There is another approach to avoid ambiguity in the behavior of players when any continuation of the game gives the same payoffs. Such an approach was proposed in. It realizes the following idea: in a given personal position the player moving in this position randomizes the alternatives yielding the same payoffs with equal probabilities. It can be easily proved that such behavior will also form a subgame perfect Nash equilibrium (not only in the case the randomization is made with equal probabilities, but also if it is made with arbitrary probability distribution over the alternatives yielding the same payoff).

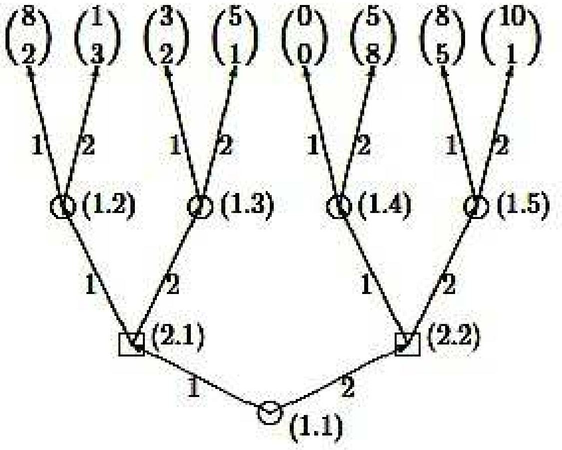
Absolute equilibrium (subgame-perfect)

For instance let us evaluate an indifferent equilibrium in the game from Example (8). In this example the alternatives 1 and 2 in position (1.2) will be chosen with probabilities $(\frac{1}{2}, \frac{1}{2})$, and in the same manner the alternatives 1 and 2 in position (1.3). In this case the Nash equilibrium (indifferent) will give the same payoffs (2,1) as favorable equilibrium. Consider now another example.

Here Player I moves in positions $\{(1.1), (1.2), (1.3)\}$, and Player II in positions $\{(2.1), (2.2)\}$. In this game in "favorable" equilibrium Player I chooses alternative 2 in position (1.2), and Nash equilibrium will give the payoffs (3, 6). In unfavorable equilibrium Player I chooses in position (1.2) alternative 1, and Nash equilibrium will give the payoffs (4, 3). In indifferent equilibrium Player I will choose alternatives 1, 2 with probabilities $(\frac{1}{2}, \frac{1}{2})$ in position (1.2). The payoffs in indifferent equilibrium will be (3, 4). In this example as well as in Example 5 there are infinite many subgame perfect Nash equilibrium, since Player I in position (1.2) can choose the alternatives 1 and 2 with any probability $p = (p_1, p_2)$, $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$, and all this mixed behavior in position (1.2) will be part of some subgame perfect Nash equilibrium.

Penalty strategies

One can prove the existence of absolute (Nash) equilibria in multistage games with perfect information on a finite graph tree. However, the investigation of particular games of this class may reveal the whole family of equilibria whose truncations are not necessarily equilibria in all subgames of the original game. Among such equilibria are equilibria in penalty strategies. We shall demonstrate this with the examples below. *Example 11.* Suppose the game Γ proceeds on the graph depicted in Figure. The set $N = \{1, 2\}$ is made up of two players. In Figure, as an Example 8, the circles represent the nodes making up the set X_1 and the blocks represent the set X_2. The nodes of the graph are designated by double indices and the arcs by single indices.


(8 2)
(1 3)
(3 2)
(5 1)
(0 0)
(5 8)
(8 5)
(10 1)
1
2
1
2
1
2
1
2
(1.2)
(1.3)
(1.4)
(1.5)
1
2
1
2
(2.1)
(2.2)
1
2
(1.1)

Penalty strategies

It can be easily seen that the situation $u_1^* = (1, 1, 2, 2, 2)$, $u_2^* = (1, 1)$ is an absolute equilibrium in the game Γ. In this case, the payoffs to players are 8 and 2 units, respectively. Now consider the situation $\overline{u}_1 = (2, 1, 2, 1, 2)$, $\overline{u}_2 = (2, 2)$. In this situation the payoffs to players respectively are 10 and 1, and thus Player 1 receives a greater amount than in the situation (u_1^*, u_2^*). The situation $(\overline{u}_1, \overline{u}_2)$ is equilibrium in the game Γ but not absolute equilibrium. In fact, in the subgame $\Gamma_{1.4}$ the truncation of the strategy $\overline{u}_1$ tells Player 1 to choose the left-hand arc, which is not optimal for him in position 1.4. Such an action taken by Player 1 in position 1.4 can be interpreted as a "penalty" threat to Player 2 if he avoids Player 1's desirable choice of arc 2 in position 2.2, thereby depriving Player 1 of the maximum payoff of 10 units. But this "penalty" threat is unlikely to be treated as valid, because the penalizer (Player 1) may lose in this case 5 units (acting nonoptimally).

Repeated games and equilibrium in punishment (penalty) strategies.

Consider the "Prisoners dilemma" game G (Example (8)).

$$G = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \left[\begin{array}{cc} (5,5) & (0,10) \\ (10,0) & (1,1) \end{array} \right] \end{array}.$$

Here the unique Nash equilibrium is (α_2, β_2) with payoff (1,1) which is dominated by pareto-optimal outcome (α_1, β_1) with payoff (5,5). Assume that the game G is played infinitely many times, and after each play (stage) players have the information about previous choices of each other. As a result we get infinite steam of payoffs, and there is a common discount factor $0 < \delta < 1$, such that the payoff of each player in the infinitely repeated game is defined as

$$\sum_{t=0}^{\infty} \delta^t \times (\text{payoff from t-th play of the stage game}).$$

The notation stage game is used for one shot game G in order to distinguish this game from the infinitely repeated game.

Repeated games and equilibrium in punishment (penalty) strategies.

As it was explained earlier a strategy of player prescribes at each moment an action (behavior) of player or mixed behavior for each sequence of length t of stage strategies $\{(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_2, \beta_1), (\alpha_2, \beta_2)\}$. Denote the just defined game by G_δ^∞. As solution we shall try to find the subgame perfect Nash equilibrium. It is clear that each subgame starting from the stage t $G_\delta^\infty(t)$ coincides with the initial game G_δ^∞.

Repeated games and equilibrium in punishment (penalty) strategies.

There are many subgame perfect Nash equilibrium (NE) in G_δ^∞.

A One is trivial: play in each stage game NE (α_2, β_2). If both players use (α_2, β_2), the payoff of each player will be

$$\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}.$$

B In stage game G an each stage t such that on the previous stages only (α_1, β_1) has occurred play (α_1, β_1). Otherwise play (α_2, β_2).

If this strategies are used by both players, then they will always play (α_1, β_1) and the payoffs of both players will be

$$\sum_{t=0}^{\infty} 5\delta^t = \frac{5}{1-\delta}.$$

There is also a subgame perfect Nash equilibrium for δ large enough. It is easy to derive the conditions on δ. Suppose one of the players, say Player II deviates, this means that there exist the first time instant (first stage game) $\bar{t}$, when he chooses β_2 instead of β_1.

Repeated games and equilibrium in punishment (penalty) strategies.

Then since on the next stage $\bar{t}+1$ first player will choose α_2 till the end of the game, his payoff (of deviation) will be equal at most to

$$\sum_{t=0}^{\bar{t}-1} 5\delta^t + 10\delta^{\bar{t}} + \sum_{t=\bar{t}+1}^{\infty} 1\delta^t.$$

If player two is not deviating his payoff will be

$$\sum_{t=0}^{\infty} 5\delta^t = \frac{5}{1-\delta}.$$

The deviation will be not preferable if

$$\sum_{t=0}^{\infty} 5\delta^t > \sum_{t=0}^{\bar{t}-1} 5\delta^t + 10\delta^{\bar{t}} + \sum_{t=\bar{t}+1}^{\infty} 1\delta^t,$$

or if $\delta > 0,6$.

Thus we proved that for $\delta > 0,6$ the strategy pair constructed in B is Nash equilibrium (it can be proved that it is also a subgame perfect Nash equilibrium).